

Fourier series

Introduction

In mathematics, a Fourier series decomposes any periodic function or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines (or complex exponentials). The study of Fourier series is a branch of Fourier analysis. Fourier series were introduced by Joseph Fourier (1768–1830) for the purpose of solving the heat equation in a metal plate. The heat equation is a partial differential equation. Prior to Fourier's work, there was no known solution to the heat equation in a general situation, although particular solutions were known if the heat source behaved in a simple way, in particular, if the heat source was a sine or cosine wave. These simple solutions are now sometimes called Eigen solutions. Fourier's idea was to model a complicated heat source as a superposition (or linear combination) of simple sine and cosine waves, and to write the solution as a superposition of the corresponding Eigen solutions. This superposition or linear combination is called the Fourier series. Although the original motivation was to solve the heat equation, it later became obvious that the same

techniques could be applied to a wide array of mathematical and physical problems. The Fourier series has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, thin-walled shell theory, etc.

Fourier series is named in honour of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli. He applied this technique to find the solution of the heat equation, publishing his initial results in his 1807 *Mémoire sur la propagation de la chaleur dans les corps solides* and 1811, and publishing his *Théorie analytique de la chaleur* in 1822.

From a modern point of view, Fourier's results are somewhat informal, due to the lack of a precise notion of function and integral in the early nineteenth century. Later, Dirichlet and Riemann expressed Fourier's results with greater precision

and formality. Fourier series are used in the analysis of periodic functions. Many of the phenomena studied in engineering and science are periodic in nature eg. the current and voltage in an alternating current circuit. These periodic functions can be analysed into their constituent components (fundamentals and harmonics) by a process called Fourier analysis. We are aiming to find an approximation using trigonometric functions for various square, saw tooth, etc waveforms that occur in electronics. We do this by adding more and more trigonometric functions together. The sum of these special trigonometric functions is called the Fourier series.

The Fourier series is an infinite series expansion involving trigonometric functions, i.e. the function defined in the interval $[-T, T]$ can be expressed in series form of sinusoidal wave. A periodic waveform $f(x)$ of period $p = 2T$ has a Fourier Series given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right)$$

$$a_0 = \frac{1}{T} \int_{-T}^T f(x) dx, \quad a_n = \frac{1}{T} \int_{-T}^T f(x) \cos\left(\frac{n\pi x}{T}\right) dx,$$

$$b_n = \frac{1}{T} \int_{-T}^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx.$$

Dirichlet Conditions

Any periodic waveform of period $p = 2T$, can be expressed in a Fourier series provided that

- (a) It has a finite number of discontinuities within the period $2T$;
- (b) It has a finite average value in the period $2T$;
- (c) It has a finite number of positive and negative maxima and minima.

When these conditions, called the Dirichlet conditions, are satisfied, the Fourier series for the function $f(t)$ exists.

Each of the examples in this chapter obey the Dirichlet Conditions and so the Fourier series exists.

Example 1:

$$f(x) = \begin{cases} 0 & -4 \leq x < 0 \\ 5 & 0 \leq x < 4 \end{cases}, \text{ expand in Fourier series}$$

Solution

This function is neither even nor odd, therefore

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right), \quad T=4$$

$$a_0 = \frac{1}{4} \int_{-4}^4 f(x) dx = \frac{1}{4} \int_0^4 5 dx = 5$$

$$a_n = \frac{1}{4} \int_{-4}^4 f(x) \cos\left(\frac{n\pi x}{4}\right) dx = \frac{1}{4} \int_0^4 5 \cos\left(\frac{n\pi x}{4}\right) dx = 0$$

$$b_n = \frac{1}{4} \int_{-4}^4 f(x) \sin\left(\frac{n\pi x}{4}\right) dx = \frac{1}{4} \int_0^4 5 \sin\left(\frac{n\pi x}{4}\right) dx = \frac{5}{n\pi} (1 - \cos n\pi)$$

If n is even, then $b_n = 0$, and if n is odd, then $b_n = \frac{10}{n\pi}$, thus

$$b_n = \frac{10}{(2n-1)\pi}, \text{ so } f(x) = \frac{5}{2} + \sum_{n=1}^{\infty} \frac{10}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{4}\right)$$

Example 2:

$$f(x) = x + x^2, \text{ expand in Fourier series } -2 < x < 2$$

Solution

This function is neither even nor odd, therefore

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right) \quad , T=2$$

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \int_{-2}^2 (x^2 + x) dx = \frac{1}{2} \left(\frac{x^3}{3} + \frac{x^2}{2} \right)_{-2}^2 = \frac{8}{3}$$

$$a_n = \frac{1}{2} \int_{-2}^2 (x^2 + x) \cos\left(\frac{n\pi x}{2}\right) dx = \frac{1}{2} \left[(x^2 + x) \left(\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \right) - (2x + 1) \left(-\frac{4}{n^2 \pi^2} \cos\left(\frac{n\pi x}{2}\right) \right) + 2 \left(-\frac{8}{n^3 \pi^3} \sin\left(\frac{n\pi x}{2}\right) \right) \right]_{-2}^2 = \frac{16}{n^2 \pi^2} \cos n\pi$$

$$b_n = \frac{1}{2} \int_{-2}^2 (x^2 + x) \sin\left(\frac{n\pi x}{2}\right) dx = \frac{1}{2} \left[(x^2 + x) \left(-\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right) - (2x + 1) \left(-\frac{4}{n^2 \pi^2} \sin\left(\frac{n\pi x}{2}\right) \right) + 2 \left(\frac{8}{n^3 \pi^3} \cos\left(\frac{n\pi x}{2}\right) \right) \right]_{-2}^2 = \frac{-4}{n\pi} \cos n\pi$$

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2 \pi^2} \cos n\pi \cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n\pi} \cos n\pi \sin\left(\frac{n\pi x}{2}\right)$$

Fourier series of even and odd functions

This section can make our lives a lot easier because it reduces the work required. In some of the problems that we encounter, the Fourier coefficients a_0 , a_n or b_n become zero after integration.

Finding zero coefficients in such problems is time consuming and can be avoided. With knowledge of even and odd functions, a zero coefficient may be predicted without performing the integration.

1- Fourier series of Even function

A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all values of x . The graph of an even function is always symmetrical about the y-axis (i.e. it is a mirror image).

Even function defined in the interval $[-T, T]$ expressed in Fourier series such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right), \quad -T < x < T$$

$$a_0 = \frac{2}{T} \int_0^T f(x) dx, \quad a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{n\pi x}{T}\right) dx, \quad b_n = 0$$

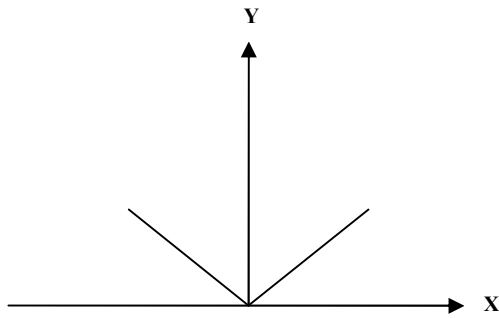
Example 3:

a) $f(x) = |x|$, $-3 < x < 3$, b) $f(x) = x^2$, $-\pi < x < \pi$

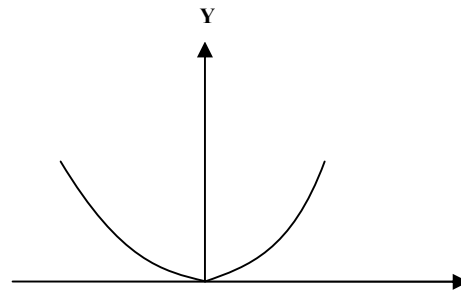
Expand in Fourier series the above functions.

Solution

The functions $f(x) = |x|$, $f(x) = x^2$ are even



$$f(x) = |x|$$



$$f(x) = x^2$$

a) $f(-x) = |-x| = |x| = f(x)$, therefore $f(x)$ is even

Hence $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right)$, $T=3$

Since $|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$, therefore

$$a_0 = \frac{2}{3} \int_0^3 f(x) dx = \frac{2}{3} \int_0^3 x dx = \left(\frac{x^2}{3}\right)_0^3 = 3$$

$$\begin{aligned} a_n &= \frac{2}{3} \int_0^3 x \cos\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3} \left[x \left(\frac{3}{n\pi} \sin\left(\frac{n\pi x}{3}\right) \right) - \left(\frac{-9}{n^2 \pi^2} \cos\left(\frac{n\pi x}{3}\right) \right) \right]_0^3 \\ &= \frac{6}{n^2 \pi^2} (\cos n\pi - 1) \end{aligned}$$

$$\text{Thus } f(x) = \frac{3}{2} + \frac{6}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{\cos n\pi - 1}{n^2} \right) \cos\left(\frac{n\pi x}{3}\right)$$

b) $f(-x) = (-x)^2 = x^2 = f(x)$, therefore $f(x)$ is even, hence

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right), \quad T = \pi, \text{ where}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \left(\frac{2x^3}{3\pi} \right)_0^{\pi} = \frac{2\pi^2}{3}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx = \frac{2}{\pi} \left[x^2 \left(\frac{\sin(nx)}{n} \right) - \frac{2x}{n^2} (-\cos(nx)) \right. \\ &\quad \left. + \frac{2}{n^3} (-\sin(nx)) \right]_0^{\pi} = \frac{4}{n^2} (\cos n\pi) \end{aligned}$$

$$\text{Therefore } f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left(\frac{\cos n\pi}{n^2} \right) \cos\left(\frac{n\pi x}{3}\right)$$

2- Fourier series of odd function

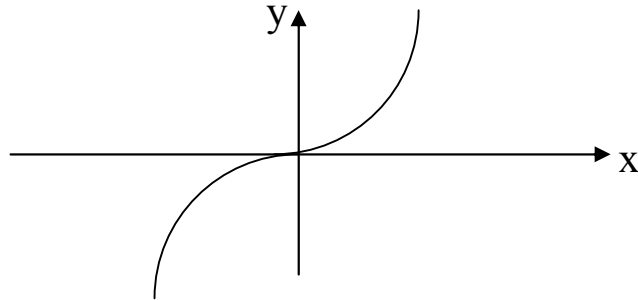
A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for all values of x . The graph of an odd function is always symmetrical about the origin.

Odd function defined in the interval $[-T, T]$ expressed in Fourier series such that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right), \text{ where } b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx ,$$

$$a_0 = a_n = 0$$

Example 4: $f(x) = x^3$, $-1 < x < 1$, expand in Fourier series



Solution

$f(-x) = (-x)^3 = -x^3 = -f(x)$, therefore $f(x)$ is odd

Hence $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right)$, $T = 1$

$$b_n = \frac{2}{1} \int_0^1 x^3 \sin(n\pi x) dx = 2 \left[x^3 \left(\frac{-\cos(n\pi x)}{n\pi} \right) - 3x^2 \left(\frac{-\sin(n\pi x)}{n^2\pi^2} \right) \right. \\ \left. + 6x \left(\frac{\cos(n\pi x)}{n^3\pi^3} \right) - 6 \left(\frac{\sin(n\pi x)}{n^4\pi^4} \right) \right]_0^1 = \frac{6\cos(n\pi)}{n^3\pi^3} - \frac{2\cos(n\pi)}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{6\cos(n\pi)}{n^3\pi^3} - \frac{2\cos(n\pi)}{n\pi} \right] \sin(n\pi x)$$

Study the following relations that may be needed for your integrals such that

$$\cos x \cos y = \frac{1}{2} (\cos(x-y) + \cos(x+y))$$

$$\sin x \sin y = \frac{1}{2} (\cos(x-y) - \cos(x+y))$$

$$\sin x \cos y = \frac{1}{2} (\sin(x-y) + \sin(x+y))$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

3- Half range cosine

An even function can be expanded using half its range from 0 to T or from $-T$ to 0. That is, the range of integration = T. The Fourier series of the half range even function is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right), \quad 0 < x < T$$

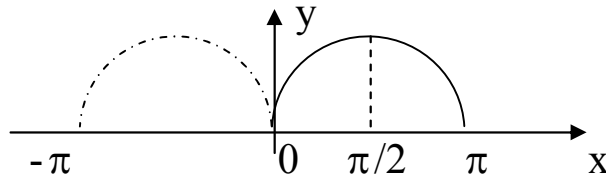
$$a_0 = \frac{2}{T} \int_0^T f(x) dx, \quad a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{n\pi x}{T}\right) dx, \quad b_n = 0$$

Example 5: Expand in half range cosine the function

a) $f(x) = \sin x$, $0 < x < \pi$, b) $f(x) = x$, $0 < x < 1$

Solution

a) We have to extend $\sin x$ $-\pi < x < 0$ to form even function so that it will be similar to $|\sin x|$, $-\pi < x < \pi$, $T = \pi$



$$f(x) = \sin x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right), \quad 0 < x < \pi$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \left(\frac{-2\cos x}{\pi} \right)_0^{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos\left(\frac{n\pi x}{\pi}\right) dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos(nx) dx, \quad n > 1$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(1-n)x + \sin(1+n)x] dx$$

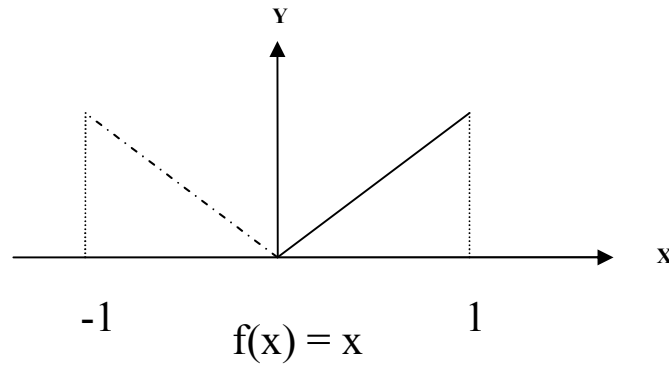
$$= \frac{1}{\pi} \left(\frac{-\cos(1-n)x}{1-n} + \frac{-\cos(1+n)x}{1+n} \right)$$

$$= \frac{2\cos(n\pi)}{\pi(1-n^2)} + \frac{2}{\pi(1-n^2)} = \frac{2(\cos(n\pi)+1)}{\pi(1-n^2)}$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x \, dx = \frac{2}{\pi} \int_0^{\pi} \sin 2x \, dx = \left(\frac{-\cos 2x}{2\pi} \right)_0^{\pi} = 0$$

$$\text{Thus } f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2(\cos(n\pi)+1)}{\pi(1-n^2)} \cos(nx)$$

b) We have to extend x , $-1 < x < 0$ to form even function so that it will be similar to $|x|$, $-1 < x < 1$, $T = 1$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right), \quad a_0 = \frac{2}{1} \int_0^1 x \, dx = (x^2)_0^1 = 1$$

$$a_n = \frac{2}{1} \int_0^1 x \cos\left(\frac{n\pi x}{1}\right) dx = 2 \left(x \sin\left(\frac{n\pi x}{n\pi}\right) - \left(\frac{-\cos(n\pi x)}{n^2 \pi^2} \right) \right)_0^1$$

$$= 2 \left(\frac{\cos(n\pi x) - 1}{n^2 \pi^2} \right), \text{ so } f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2 \left(\frac{\cos(n\pi x) - 1}{n^2 \pi^2} \right) \cos(n\pi x)$$

4- Half range sine

An odd function can be expanded using half its range from 0 to T, i.e. the range of integration = T. The Fourier series of the odd function is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right), \text{ where } b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx ,$$

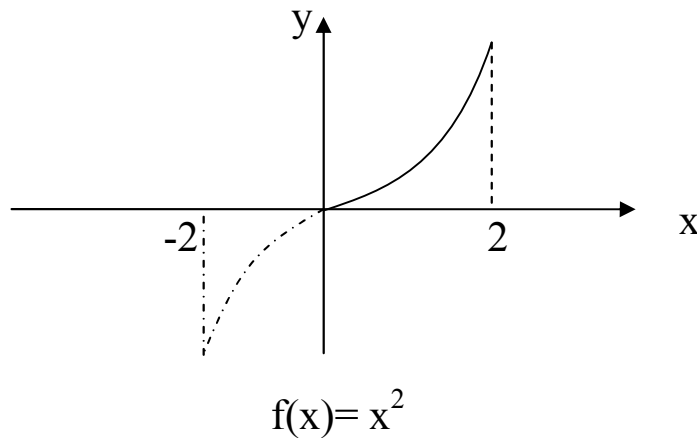
$$a_0 = a_n = 0 \quad , \quad 0 < x < T$$

Example 6: Expand in half range sine the following functions

a) $f(x) = x^2 \quad 0 < x < 2$, b) $f(x) = \cos x \quad 0 < x < \pi$

Solution

a) We have to extend x^2 , $-2 < x < 0$ to form odd function, $T = 2$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right), \text{ where } b_n = \frac{2}{T} \int_0^T x^2 \sin\left(\frac{n\pi x}{T}\right) dx$$

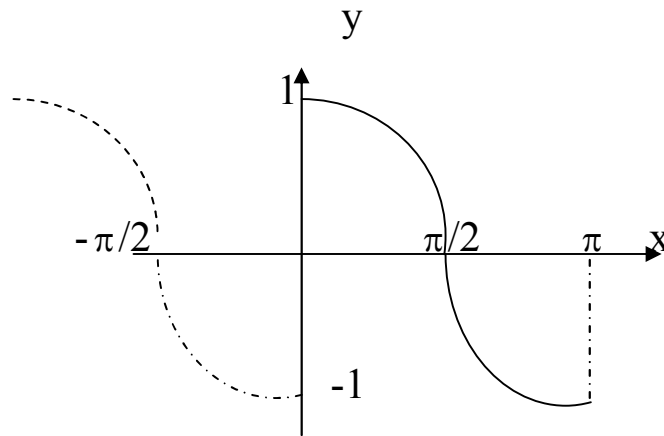
$$= \left[x^2 \left(-\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right) - 2x \left(-\frac{4}{n^2 \pi^2} \sin\left(\frac{n\pi x}{2}\right) \right) \right.$$

$$\left. + 2 \left(\frac{8}{n^3 \pi^3} \cos\left(\frac{n\pi x}{2}\right) \right) \right]_0^2 = -\frac{8}{n\pi} \cos(n\pi) + \frac{16(\cos(n\pi) - 1)}{n^3 \pi^3}$$

$$\text{Thus } f(x) = \sum_{n=1}^{\infty} \left[-\frac{8}{n\pi} \cos(n\pi) + \frac{16(\cos(n\pi) - 1)}{n^3 \pi^3} \right] \sin\left(\frac{n\pi x}{2}\right)$$

b) We have to extend $\cos x$, $-\pi < x < 0$ to form odd function

$$T = \pi$$



$$f(x) = \cos x$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right),$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \cos x \sin\left(\frac{n\pi x}{\pi}\right) dx = \frac{1}{\pi} \int_0^{\pi} (\sin(n-1)x + \sin(n+1)x) dx \\ &= \frac{1}{\pi} \left(\frac{-\cos(n-1)x}{n-1} + \frac{-\cos(n+1)x}{n+1} \right) \Big|_0^{\pi} = \frac{2n\cos(n\pi)}{\pi(n^2-1)} + \frac{2n}{\pi(n^2-1)} \\ &= \frac{2n(\cos(n\pi)+1)}{\pi(n^2-1)}, n > 1 \end{aligned}$$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(x) dx = \frac{1}{\pi} \int_0^{\pi} \sin(2x) dx = \left(\frac{-\cos 2x}{2\pi} \right) \Big|_0^{\pi} = 0$$

$$\text{Thus } f(x) = \sum_{n=2}^{\infty} \left[\frac{2n(\cos(n\pi)+1)}{\pi(n^2-1)} \right] \sin\left(\frac{n\pi x}{T}\right)$$

Harmonic analysis

Refer to full range Fourier series in the interval $[-\pi, \pi]$

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right) \\ &= \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + \dots + b_1 \sin(x) \\ &\quad + b_2 \sin(2x) + b_3 \sin(3x) + \dots \end{aligned}$$

We can re-arrange this series and write it as:

$$f(x) = \frac{a_0}{2} + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) \\ + a_3 \cos(3x) + b_3 \sin(3x) + \dots$$

The term $a_1 \cos(x) + b_1 \sin(x)$ is known as the fundamental, the term $a_2 \cos(2x) + b_2 \sin(2x)$ is called the second harmonic and the term $a_3 \cos(3x) + b_3 \sin(3x)$ is called the third harmonic, etc.

1- Odd harmonic

The Fourier series will contain odd harmonics if $f(x+T) = -f(x)$, where the period $= 2T$. In this case, the Fourier expansion in the interval $[-\pi, \pi]$ will be of the form:

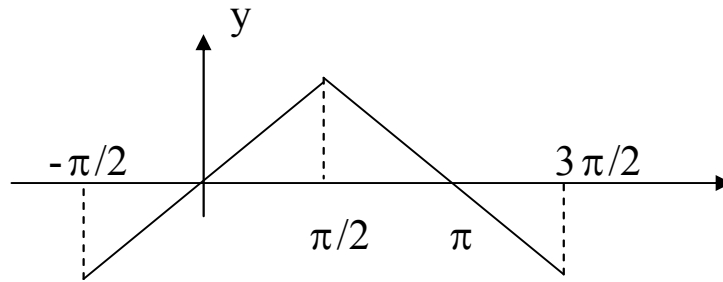
$(a_1 \cos(x) + b_1 \sin(x)) + (a_3 \cos(3x) + b_3 \sin(3x)) + (a_5 \cos(5x) + b_5 \sin(5x)) + \dots$ and all of the harmonics are odd. In other word, the expansion will be

$$f(x) = \sum_{n=1}^{\infty} a_{2n-1} \cos((2n-1)x) + \sum_{n=1}^{\infty} b_{2n-1} \sin((2n-1)x)$$

$$a_{2n-1} = \frac{2}{T} \int_0^T f(x) \cos((2n-1)x) dx, \quad b_{2n-1} = \frac{2}{T} \int_0^T f(x) \sin((2n-1)x) dx$$

Example 7: Expand in Fourier series

$f(x) = x$, $-\pi/2 < x < \pi/2$, $f(x + \pi) = -f(x)$, period $= 2\pi$

**Solution**

Since $f(x + \pi) = -f(x)$, therefore the expansion will be odd

harmonic such that $f(x) = \begin{cases} x, & -\pi/2 \leq x \leq \pi/2 \\ -(x - \pi), & \pi/2 < x \leq 3\pi/2 \end{cases}$

Thus $f(x) = \sum_{n=1}^{\infty} a_{2n-1} \cos(2n-1)x + \sum_{n=1}^{\infty} b_{2n-1} \sin(2n-1)x$

$$\begin{aligned} a_{2n-1} &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \cos(2n-1)x \, dx \\ &= \frac{2}{\pi} \left(x \left(\frac{\sin(2n-1)x}{2n-1} \right) - \left(\frac{-\cos(2n-1)x}{(2n-1)^2} \right) \right) \Bigg|_{-\pi/2}^{\pi/2} = 0 \end{aligned}$$

$$\begin{aligned}
b_{2n-1} &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(2n-1)x \, dx \\
&= \frac{2}{\pi} \left(x \left(\frac{-\cos(2n-1)x}{2n-1} \right) - \left(\frac{-\sin(2n-1)x}{(2n-1)^2} \right) \right) \Big|_{-\pi/2}^{\pi/2} \\
&= -\frac{4}{\pi(2n-1)^2} \cos n\pi, \text{ therefore}
\end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty} b_{2n-1} \sin(2n-1)x = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{(2n-1)^2} \sin(2n-1)x$$

The above example can be solved by other way such that

$$a_0 = a_{2n} = a_{2n-1} = b_{2n} = 0 \text{ and}$$

$$\begin{aligned}
b_{2n-1} &= \frac{4}{\pi} \int_0^{\pi/2} x \sin(2n-1)x \, dx \\
&= \frac{4}{\pi} \left(x \left(\frac{-\cos(2n-1)x}{2n-1} \right) - \left(\frac{-\sin(2n-1)x}{(2n-1)^2} \right) \right) \Big|_0^{\pi/2} \\
&= -\frac{4}{\pi(2n-1)^2} \cos n\pi
\end{aligned}$$

This expansion is called odd sine harmonic defined in quarter of the period.

2- Even harmonic

The Fourier series will contain even harmonics if $f(x + T) = f(x)$, where the period $= 2T$. In this case, the Fourier expansion in the interval $[-\pi, \pi]$ will be of the form:

$\frac{a_0}{2} + (a_2 \cos(2x) + b_2 \sin(2x)) + (a_4 \cos(4x) + b_4 \sin(4x)) + (a_6 \cos(6x) + b_6 \sin(6x)) + \dots$ and all of the harmonics are even. In other word, the expansion will be

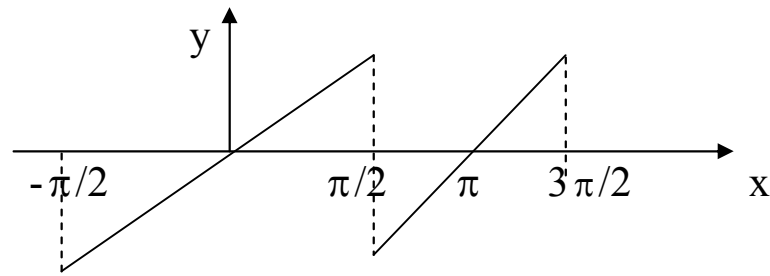
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos(2n)x + \sum_{n=1}^{\infty} b_{2n} \sin(2n)x$$

$$a_0 = \frac{2}{T} \int_0^T f(x) dx, \quad a_{2n} = \frac{2}{T} \int_0^T f(x) \cos(2n)x dx,$$

$$b_{2n-1} = \frac{2}{T} \int_0^T f(x) \sin(2n)x dx$$

Example 8: Expand in Fourier series

$f(x) = x$, $-\pi/2 < x < \pi/2$, $f(x + \pi) = f(x)$, period $= 2\pi$



Since $f(x + \pi) = f(x)$, therefore the expansion will be even

harmonic such that $f(x) = \begin{cases} x, & -\pi/2 \leq x \leq \pi/2 \\ (x - \pi), & \pi/2 < x \leq 3\pi/2 \end{cases}$

Thus $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos(2n)x + \sum_{n=1}^{\infty} b_{2n} \sin(2n)x$

$$a_0 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \, dx = 0,$$

$$\begin{aligned} a_{2n} &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \cos(2nx) \, dx \\ &= \frac{2}{\pi} \left(x \left(\frac{\sin(2nx)}{2n} \right) - \left(\frac{-\cos(2nx)}{4n^2} \right) \right) \Big|_{-\pi/2}^{\pi/2} = 0 \end{aligned}$$

$$\begin{aligned}
b_{2n} &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(2nx) dx \\
&= \frac{2}{\pi} \left(x \left(\frac{-\cos(2nx)}{2n} \right) - \left(\frac{-\sin(2nx)}{4n^2} \right) \right)_{-\pi/2}^{\pi/2} = -\frac{1}{n} \cos n\pi
\end{aligned}$$

$$\text{Therefore } f(x) = \sum_{n=1}^{\infty} b_{2n} \sin(2nx) = - \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin(2nx)$$

The above example can be solved by other way such that

$$a_0 = a_{2n} = a_{2n-1} = b_{2n-1} = 0 \text{ and}$$

$$\begin{aligned}
b_{2n} &= \frac{4}{\pi} \int_0^{\pi/2} x \sin(2nx) dx \\
&= \frac{4}{\pi} \left(x \left(\frac{-\cos(2nx)}{2n} \right) - \left(\frac{-\sin(2nx)}{4n^2} \right) \right)_0^{\pi/2} = -\frac{1}{n} \cos n\pi
\end{aligned}$$

This expansion is called even sine harmonic defined in quarter of the period.

3- Even cosine harmonic

The Fourier series will contain even cosine harmonics if $f(x + T) = f(x)$, and $f(x) = f(-x)$ where the period = $2T$. It is even harmonic but without sine components.

In this case, the Fourier expansion will be of the form

$\frac{a_0}{2} + a_2 \cos(\frac{2\pi x}{T}) + a_4 \cos(\frac{4\pi x}{T}) + \dots$ and all of the harmonics

are even. In other word, the expansion will be

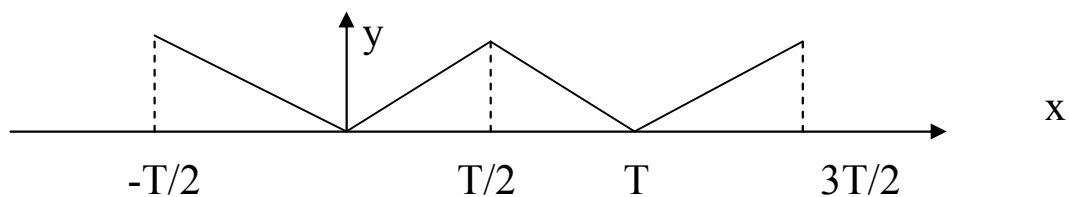
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos(\frac{2n\pi x}{T})$$

$$a_0 = \frac{4}{T} \int_0^{T/2} f(x) dx, \quad a_{2n} = \frac{4}{T} \int_0^{T/2} f(x) \cos(\frac{2n\pi x}{T}) dx, \quad b_{2n} = 0$$

Example 9: Sketch $f(x) = x$ $0 < x < T/2$ and $f(x + T) = f(x)$, and $f(x) = f(-x)$ where the period $= 2T$.

Solution

It is clear that the expansion will be even cosine harmonic



4- Even sine harmonic

The Fourier series will contain even sine harmonics if $f(x + T) = f(x)$, and $-f(x) = f(-x)$ where the period $= 2T$. It is even harmonic but without a_n components, i.e. $a_0 = a_{2n} = 0$.

In this case, the Fourier expansion will be of the form

$$b_2 \sin\left(\frac{2\pi x}{T}\right) + b_4 \sin\left(\frac{4\pi x}{T}\right) + b_6 \sin\left(\frac{6\pi x}{T}\right) + \dots \text{ and all of the}$$

harmonics are even. In other word, the expansion will be

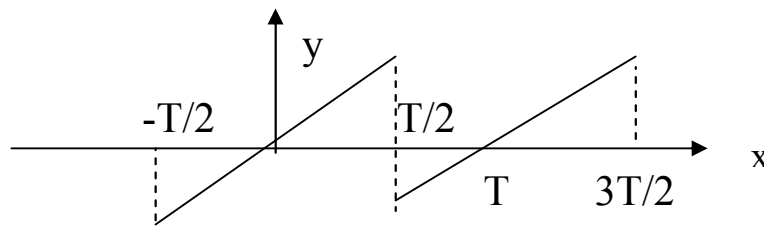
$$f(x) = \sum_{n=1}^{\infty} b_{2n} \sin\left(\frac{2n\pi x}{T}\right)$$

$$a_0 = a_{2n} = 0, \quad b_{2n} = \frac{4}{T} \int_0^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx$$

Example 10: Sketch $f(x) = x$ $0 < x < T/2$ and $f(x + T) = f(x)$, and $f(x) = -f(-x)$ where the period = $2T$.

Solution

It is clear that the expansion will be even sine harmonic



5- Odd cosine harmonic

The Fourier series will contain odd cosine harmonics if $f(x+T) = -f(x)$, and $f(x) = f(-x)$ where the period = $2T$. It is odd harmonic but without sine components.

In this case, the Fourier expansion will be of the form

$$a_1 \cos\left(\frac{\pi x}{T}\right) + a_3 \cos\left(\frac{3\pi x}{T}\right) + a_5 \cos\left(\frac{5\pi x}{T}\right) + \dots \text{ and all of the}$$

harmonics are odd. In other word, the expansion will be

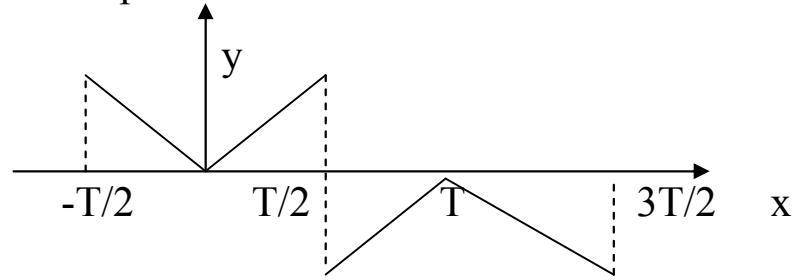
$$f(x) = \sum_{n=1}^{\infty} a_{2n-1} \cos\left(\frac{(2n-1)\pi x}{T}\right)$$

$$b_{2n-1} = 0, \quad a_{2n-1} = \frac{4}{T} \int_0^{T/2} f(x) \cos\left(\frac{(2n-1)\pi x}{T}\right) dx,$$

Example 11: Sketch $f(x) = x$, $0 < x < T/2$, $f(x + T) = -f(x)$, and $f(x) = f(-x)$ where the period = $2T$.

Solution

It is clear that the expansion will be odd cosine harmonic



6- Odd sine harmonic

The Fourier series will contain odd sine harmonics if $f(x+T) = -f(x)$, and $f(x) = -f(-x)$ where the period $= 2T$. It is odd harmonic but without cosine components.

In this case, the Fourier expansion will be of the form

$$b_1 \sin\left(\frac{\pi x}{T}\right) + b_3 \sin\left(\frac{3\pi x}{T}\right) + b_5 \sin\left(\frac{5\pi x}{T}\right) + \dots \text{ and all of the}$$

harmonics are odd. In other word, the expansion will be

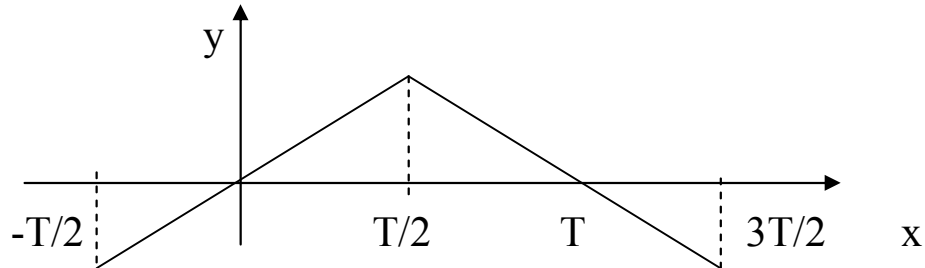
$$f(x) = \sum_{n=1}^{\infty} b_{2n-1} \sin\left(\frac{(2n-1)\pi x}{T}\right)$$

$$a_{2n-1} = 0, \quad b_{2n-1} = \frac{4}{T} \int_0^{T/2} f(x) \sin\left(\frac{(2n-1)\pi x}{T}\right) dx .$$

Example 12: Sketch $f(x) = x$ $0 < x < T/2$ and $f(x + T) = -f(x)$, and $f(x) = -f(-x)$ where the period $= 2T$.

Solution

It is clear that the expansion will be odd sine harmonic



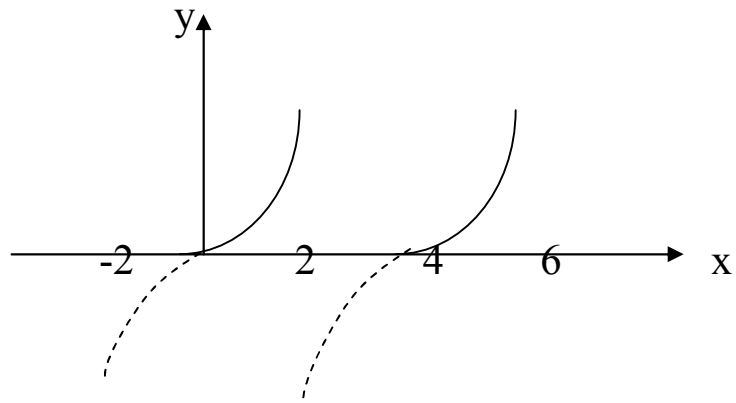
Example 13: Expand the function $f(x) = x^2$, $0 < x < 2$, $f(x+4) = f(x)$ in

- a) Even sine harmonic b) Odd cosine harmonic

Solution: Since $T = 4$, therefore

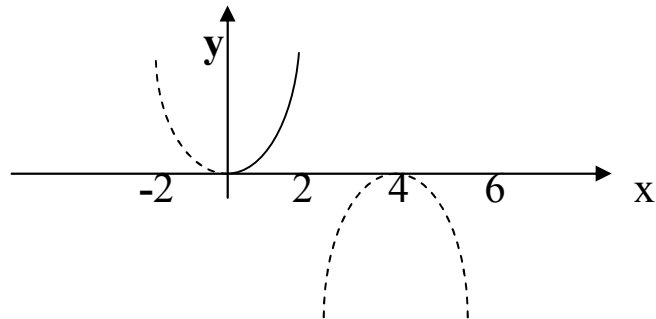
a) $f(x) = \sum_{n=1}^{\infty} b_{2n} \sin\left(\frac{2n\pi x}{T}\right)$, where $a_0 = a_{2n} = 0$, and

$$\begin{aligned} b_{2n} &= \frac{4}{T} \int_0^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx = \frac{4}{4} \int_0^2 x^2 \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \left(x^2 \left(-\frac{2 \cos\left(\frac{n\pi x}{2}\right)}{n\pi} \right) - 2x \left(-\frac{4 \sin\left(\frac{n\pi x}{2}\right)}{n^2 \pi^2} \right) + 2 \left(\frac{8 \cos\left(\frac{n\pi x}{2}\right)}{n^3 \pi^3} \right) \right) \Bigg|_0^2 \\ &= \frac{-8}{n\pi} \cos n\pi + \frac{16(\cos(n\pi)-1)}{n^3 \pi^3} \end{aligned}$$



b) $f(x) = \sum_{n=1}^{\infty} a_{2n-1} \cos\left(\frac{(2n-1)\pi x}{T}\right)$, where $b_{2n-1} = 0$, and

$$\begin{aligned} a_{2n-1} &= \frac{4}{T} \int_0^{T/2} f(x) \cos\left(\frac{(2n-1)\pi x}{T}\right) dx = \frac{4}{4} \int_0^2 x^2 \cos\left(\frac{(2n-1)\pi x}{4}\right) dx \\ &= \left(x^2 \left(-\frac{4 \sin\left(\frac{(2n-1)\pi x}{4}\right) \pi x}{(2n-1)\pi} \right) - 2x \left(-\frac{16 \cos\left(\frac{(2n-1)\pi x}{4}\right) \pi x}{(2n-1)^2 \pi^2} \right) + \right. \\ &\quad \left. 2 \left(-\frac{64 \sin\left(\frac{(2n-1)\pi x}{4}\right) \pi x}{(2n-1)^3 \pi^3} \right) \right) \Big|_0^2 = \frac{-16 \cos(n\pi) x}{(2n-1)\pi} + \frac{128 \cos(n\pi) x}{(2n-1)^3 \pi^3} \end{aligned}$$



Parseval's theorem

Consider the function $f(x)$ expressed as full range Fourier series in the range $-T < x < T$, such that $f(x)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right).$$

Then Parseval's formula is expressed by:

$$\frac{1}{T} \int_{-T}^T (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Example 14: Expand $f(x) = x^2$ in Fourier series, $-\pi < x < \pi$,

then find $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution:

Since $f(x)$ is even, thus $b_n = 0$, $a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \left(\frac{2x^3}{3\pi}\right)_0^{\pi} = \frac{2\pi^2}{3}$

and $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx = \frac{4}{n^2} (\cos n\pi)$ (solved above)

Thus $f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(\cos n\pi)}{n^2} \cos\left(\frac{n\pi x}{3}\right)$

By Parseval's theorem $\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{x^5}{5\pi} \Big|_{-\pi}^{\pi} = \frac{2\pi^4}{5} = \frac{\pi^4}{18} +$

$$16 \sum_{n=1}^{\infty} \frac{1}{n^4}, \text{ thus } \sum_{n=1}^{\infty} \frac{1}{n^4} = 31 \pi^4 / 1440.$$

Fourier transform

In mathematics, the Fourier transform (often abbreviated F.T.) is an operation that transforms one complex-valued function of a real variable into another. In such applications as signal processing, the domain of the original function is typically time and is accordingly called the time domain. The domain of the new function is typically called the frequency domain, and the new function itself is called the frequency domain representation of the original function. It describes which frequencies are present in the original function. This is analogous to describing a musical chord in terms of the individual notes being played. In effect, the Fourier transform decomposes a function into oscillatory functions. The term Fourier transforms refers both to the frequency domain representation of a function, and to the process or formula

that "transforms" one function into the other. The Fourier transform and its generalizations are the subject of Fourier analysis. In this specific case, both the time and frequency domains are unbounded linear continua. It is possible to define the Fourier transform of a function of several variables, which is important for instance in the physical study of wave motion and optics. It is also possible to generalize the Fourier transform on discrete structures such as finite groups. The efficient computation of such structures, by fast Fourier transform, is essential for high-speed computing.

Fourier transform can be expressed if the function is neither even nor odd as follows:

$$F_{\alpha}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

Example 15: Find Fourier transform for e^{-x} , $x > 0$

Solution

$$\begin{aligned} F_{\alpha}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(1-i\alpha)x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-(1-i\alpha)x}}{-(1-i\alpha)} \right)_0^{\infty} = \frac{1}{\sqrt{2\pi}(1-i\alpha)} \\ &= \frac{1+i\alpha}{\sqrt{2\pi}(1+\alpha^2)} \end{aligned}$$

Fourier transform can be expressed if the function is even as follows:

$$F_c(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x dx$$

It is called Fourier Cosine transform, while Fourier sine equal zero.

Fourier transform can be expressed if the function is odd as

follows:
$$F_s(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \sin \alpha x dx$$

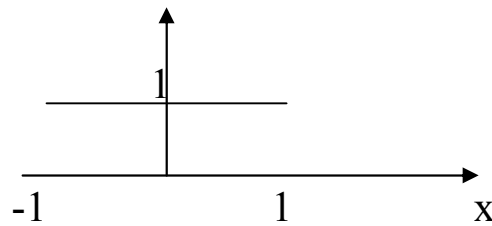
It is called Fourier sine transform, while Fourier Cosine equal zero.

Example 16 : Find Fourier transform for $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

Solution

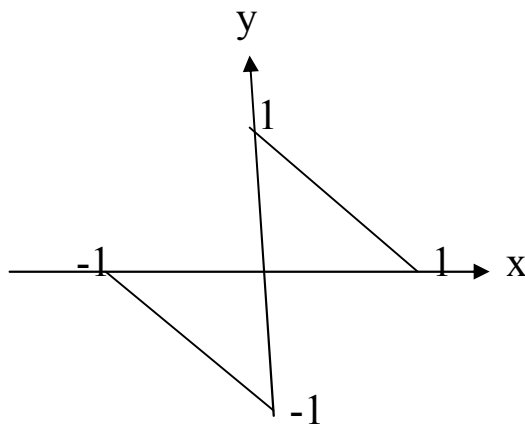
Since this function is even, therefore there is only Fourier Cosine transform such that

$$F_c(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x \, dx = \sqrt{2/\pi} \int_0^1 \cos \alpha x \, dx = \sqrt{2/\pi} \frac{\sin \alpha}{\alpha}$$



Example 17: Find Fourier transform for

$$f(x) = \begin{cases} -x+1 & 0 < x < 1 \\ -x-1 & -1 < x < 0 \end{cases}$$



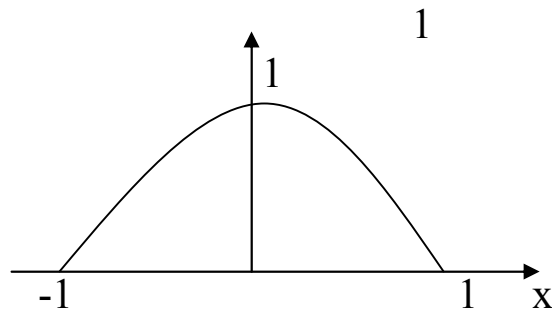
Solution

Since this function is odd, therefore there is only Fourier sine transform such

$$\begin{aligned}\text{that } F_s(\alpha) &= \sqrt{2/\pi} \int_0^{\infty} f(x) \sin \alpha x \, dx = \sqrt{2/\pi} \int_0^1 (1-x) \sin \alpha x \, dx \\ &= \sqrt{2/\pi} \left[(1-x) \left(\frac{-\cos \alpha x}{\alpha} \right) - (-1) \left(\frac{-\sin \alpha x}{\alpha^2} \right) \right]_0^1 = \sqrt{2/\pi} \left[\frac{\cos \alpha}{\alpha} - \frac{\sin \alpha}{\alpha^2} \right] \\ &= \sqrt{2/\pi} \left[\frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2} \right]\end{aligned}$$

Example 18: Find Fourier transform for

$$f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



Solution

Since this function is even, therefore there is only Fourier Cosine transform such

$$\begin{aligned}\text{that } F_c(\alpha) &= \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x \, dx = \sqrt{2/\pi} \int_0^1 (1-x^2) \cos \alpha x \, dx \\&= \sqrt{2/\pi} \left[(1-x^2) \left(\frac{\sin \alpha x}{\alpha} \right) - (-2x) \left(\frac{-\cos \alpha x}{\alpha^2} \right) + (-2) \left(\frac{-\sin \alpha x}{\alpha^3} \right) \right]_0^1 \\&= \sqrt{2/\pi} \left[\frac{-2 \cos \alpha}{\alpha^2} + \frac{2 \sin \alpha}{\alpha^3} \right] = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin \alpha - \alpha \cos \alpha}{\alpha^3} \right]\end{aligned}$$

Fourier Integral

For a non-periodic function $f(x)$, the corresponding Fourier Integral can be written as:

$$f(x) = \int_0^{\infty} (a(\alpha) \cos \alpha x + b(\alpha) \sin \alpha x) \, d\alpha$$

$$a(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx \quad \text{and} \quad b(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx$$

If $f(x)$ is even, then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(x) \cos \alpha x \, dx \right) \cos \alpha x \, d\alpha$$

If $f(x)$ is odd, then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(x) \sin \alpha x \, dx \right) \sin \alpha x \, d\alpha$$

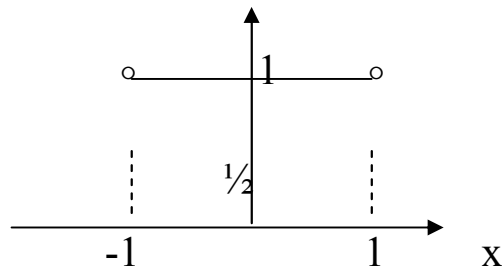
The above two equations can be written in the form

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \alpha x \, dx \right) \cos \alpha x \, d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha x \, d\alpha \quad \text{is even function.} \end{aligned}$$

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \alpha x \, dx \right) \sin \alpha x \, d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\alpha) \sin \alpha x \, d\alpha \quad \text{is odd function.} \end{aligned}$$

Example 19: Find Fourier integral for $f(x) = \begin{cases} 1 & |x| < 1 \\ 1/2 & |x| = 1 \\ 0 & |x| > 1 \end{cases}$,

deduce $\int_0^{\infty} \frac{\sin x}{x} \, dx$



Solution

Since the function $f(x)$ is even, therefore

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha x \, d\alpha$$

$$F_c(\alpha) = \sqrt{2/\pi} \int_0^1 \cos \alpha x \, dx = \sqrt{2/\pi} \left(\frac{\sin \alpha x}{\alpha} \right)_0^1 = \sqrt{2/\pi} \left(\frac{\sin \alpha}{\alpha} \right)$$

Therefore

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha x \, d\alpha = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{2/\pi} \left(\frac{\sin \alpha}{\alpha} \right) \cos \alpha x \, d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \alpha \cos \alpha x}{\alpha} \right) d\alpha \end{aligned}$$

$$\text{Therefore } \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \alpha \cos \alpha x}{\alpha} \right) d\alpha = \begin{cases} 1 & |x| < 1 \\ 1/2 & |x| = 1 \\ 0 & |x| > 1 \end{cases}$$

$$\text{At } x = 0, \text{ therefore } \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha}{\alpha} d\alpha = 1 \Rightarrow \int_0^{\infty} \frac{\sin \alpha}{\alpha} d\alpha = \frac{\pi}{2}$$

$$\text{Put } \alpha = x, \text{ thus } \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Example 20: Find Fourier integral for

$$f(x) = \begin{cases} 1 & 0 < x < a \\ -1 & -a < x < 0 \\ 0 & |x| > a \end{cases}$$

Solution

Since the function $f(x)$ is odd therefore

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\alpha) \sin \alpha x \, d\alpha$$

$$F_s(\alpha) = \sqrt{2/\pi} \int_0^a \sin \alpha x \, dx = \sqrt{2/\pi} \left(\frac{-\cos \alpha x}{\alpha} \right)_0^a = \sqrt{2/\pi} \left(\frac{1 - \cos \alpha a}{\alpha} \right)$$

Therefore

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\alpha) \sin \alpha x \, d\alpha = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{2/\pi} \left(\frac{1 - \cos \alpha a}{\alpha} \right) \sin \alpha x \, d\alpha,$$

$$\text{hence } f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1 - \cos \alpha a}{\alpha} \right) \sin \alpha x \, d\alpha = \begin{cases} 1 & 0 < x < a \\ -1 & -a < x < 0 \\ 0 & |x| > a \end{cases}.$$

$$\text{At } x = 0, \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1 - \cos \alpha a}{\alpha} \right) \sin \alpha x \, d\alpha = 0 = \frac{1}{2} [f(0^+) +$$

$f(0^-)] = \frac{1}{2}[1-1]$, so Fourier integral is verified.

Complex Fourier

In an earlier module, we showed that a square wave could be expressed as a superposition of pulses. As useful as this decomposition was in this example, it does not generalize well to other periodic signals: How can a superposition of pulses equal a smooth signal like a sinusoid?

Because of the importance of sinusoids to linear systems, you might wonder whether they could be added together to represent a large number of periodic signals. You would be right and in good company as well. Euler and Gauss in particular worried about this problem, and

Jean Baptiste Fourier got the credit even though tough mathematical issues were not settled until later. They worked on what is now known as the Fourier series: representing any periodic signal as a superposition of sinusoids. But the Fourier series goes well beyond being another signal decomposition method. Rather, the Fourier series begins our journey to appreciate how a signal can be described in either the time-domain or the frequency-domain with no compromise. Let $f(x)$ be a periodic signal with period $2T$. We want to show that periodic signals, even those that have constant-valued segments like a square wave, can be expressed as sum of harmonically related sine waves: sinusoids having frequencies that are integer multiples of the fundamental frequency. Because the signal has period $2T$, the fundamental frequency is $1/2T$. The complex Fourier series

expresses the signal as a superposition of complex exponentials having frequencies $n/2T$, $n = \dots, -1, 0, 1, 2, \dots$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i(\frac{n\pi x}{T})}, \quad c_n = \frac{1}{2T} \int_{-T}^T f(x) e^{-i(\frac{n\pi x}{T})} dx, \quad -T < x < T$$

Example 21: Find complex Fourier for $f(x) = e^{-x}$, $-2 < x < 2$

Solution

Since $T = 2$, therefore

$$\begin{aligned} c_n &= \frac{1}{2T} \int_{-T}^T f(x) e^{-i(\frac{n\pi x}{T})} dx = \frac{1}{4} \int_{-2}^2 e^{-x} e^{-i(\frac{n\pi x}{2})} dx = \frac{1}{4} \int_{-2}^2 e^{-(1+\frac{in\pi}{2})x} dx \\ &= -\frac{1}{2(2+in\pi)} (e^{-(1+\frac{in\pi}{2})x}) \Big|_{-2}^2 = \frac{1}{2(2+in\pi)} (e^{(2+in\pi)} - e^{-(2+in\pi)}) = \\ &= \frac{i}{(2+in\pi)} \sin(2+in\pi) = -\frac{1}{(2+in\pi)} (\cos 2 \sinh n\pi - i \sin 2 \cosh n\pi) \end{aligned}$$

Problems

1) Expand in Fourier series the following functions

a) $f(x) = x + x^2$, $-2 < x < 2$

b) $f(x) = |x|$, $-3 < x < 3$

c) $f(x) = x^2$, $-\pi < x < \pi$, then deduce the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$d) f(x) = \begin{cases} \pi/2 + x, & -\pi \leq x \leq 0 \\ \pi/2 - x, & 0 < x \leq \pi \end{cases}, \text{ then deduce the sum}$$

$$\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

$$e) f(x) = |\sin x|, 0 < x < 2\pi, \text{ then deduce the}$$

$$\text{sum } \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$$

$$f) f(x) = |\cos x|, -\pi < x < \pi, \text{ then deduce the sum}$$

$$\frac{1}{1^2 . 3^2} + \frac{1}{3^2 . 5^2} + \frac{1}{5^2 . 7^2} + \dots$$

$$g) f(x) = x |x|, -\pi < x < \pi$$

$$h) f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ x/\pi, & 0 < x \leq \pi \end{cases},$$

$$i) f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ -(x - \pi), & \pi < x \leq 2\pi \end{cases}$$

$$j) f(x) = \begin{cases} 0 & -1 \leq x \leq -1/2 \\ \cos 3\pi x & -1/2 < x \leq 1/2 \\ 0 & 1/2 \leq x < 1 \end{cases}, f(x+2) = f(x)$$

k) $f(x) = e^x$, $-\pi < x < \pi$, use the result to find the sum of series $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$

$$L) f(x) = \begin{cases} x, & 0 < x \leq \pi \\ x + \pi, & -\pi < x \leq 0 \end{cases}$$

2) Expand in half range cosine (cosine harmonic) the function

a) $f(x) = \sin x$, $0 < x < \pi$, b) $f(x) = x$, $0 < x < 1$,

c) $f(x) = e^x$, $0 < x < \pi$

d) $f(x) = x(\pi - x)$, $0 < x < \pi$, then deduce the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

3) Expand in half range sine (sine harmonic) the following functions

a) $f(x) = x^2$, $0 < x < 2$, b) $f(x) = \cos x$, $0 < x < \pi$,

c) $f(x) = x(\pi - x)$, $0 < x < \pi$, then deduce the sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$

4) Expand (2) and (3) in odd harmonic and in Even harmonic

5) Expand $f(x) = x$, $0 < x < 1$ in

a) Even cosine harmonic

b) Odd cosine harmonic

c) Even sine harmonic

d) Odd sine harmonic

6) Expand in complex Fourier

a) $f(x) = e^x$, $-\pi < x < \pi$, b) $f(x) = x$, $-1 < x < 1$

7) Expand the function $f(x) = x$, $0 < x < T$ in

a) Fourier cosine

b) Fourier sine

c) Odd harmonic

d) Even harmonic

e) Even Cosine harmonic

f) Even Sine harmonic

8) Find Fourier transforms of the following functions

a) $f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$, then evaluate

$$\int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos(x/2) dx$$

b) $f(x) = \begin{cases} 1 & 0 < x < a \\ -1 & -a < x < 0 \\ 0 & |x| > a \end{cases}$

9) Expand the following functions in Fourier sine series:

a) $f(x) = \begin{cases} \pi/2 - x, & 0 \leq x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi \end{cases}$,

b) $f(x) = e^x, \quad 0 < x < 1$

10) If $f(x) = \begin{cases} 0 & x \leq \pi/2 \\ \pi/2 - x & \pi/2 < x \end{cases}$, then the first Fourier

cosine coefficient equaland the second Fourier sine coefficient equal

11) Find the first Fourier cosine coefficients of the functions:

a) $f(x) = \begin{cases} x-5 & 0 < x \leq 5 \\ 0 & 5 < x \leq 10 \end{cases}$, b) $f(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1-x & 1 < x \leq 2 \end{cases}$

Model Questions

Fourier Problems

1) Expand in Fourier series the following functions

a) $f(x) = x + x^2, -2 < x < 2$

b) $f(x) = |x|, -3 < x < 3$

c) $f(x) = x^2, -\pi < x < \pi$, then deduce the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$

d) $f(x) = \begin{cases} \pi/2 + x, & -\pi \leq x \leq 0 \\ \pi/2 - x, & 0 < x \leq \pi \end{cases}$, then deduce the sum

$$\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

e) $f(x) = |\sin x|, 0 < x < 2\pi$, then deduce the sum $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

f) $f(x) = |\cos x|, -\pi < x < \pi$, then deduce the sum

$$\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \dots$$

g) $f(x) = x|x|, -\pi < x < \pi$

h) $f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ x/\pi, & 0 < x \leq \pi \end{cases}$,

i) $f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ -(x - \pi), & \pi < x \leq 2\pi \end{cases}$

j) $f(x) = \begin{cases} 0 & -1 \leq x \leq -1/2 \\ \cos 3\pi x & -1/2 < x \leq 1/2 \\ 0 & 1/2 \leq x < 1 \end{cases}$, $f(x+2) = f(x)$

k) $f(x) = e^x, -\pi < x < \pi$, use the result to find the sum of series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$L) f(x) = \begin{cases} x, & 0 < x \leq \pi \\ x + \pi, & -\pi < x \leq 0 \end{cases}$$

2) Expand in half range cosine (cosine harmonic) the function

a) $f(x) = \sin x$, $0 < x < \pi$, b) $f(x) = x$, $0 < x < 1$, c) $f(x) = e^x$, $0 < x < \pi$

d) $f(x) = x(\pi - x)$, $0 < x < \pi$, then deduce the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

3) Expand in half range sine (sine harmonic) the following functions

a) $f(x) = x^2$, $0 < x < 2$, b) $f(x) = \cos x$, $0 < x < \pi$,

c) $f(x) = x(\pi - x)$, $0 < x < \pi$, then deduce the sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$

4) Expand (2) and (3) in odd harmonic and in Even harmonic

5) Expand $f(x) = x$, $0 < x < 1$ in

- a) Even cosine harmonic b) Odd cosine harmonic
- c) Even sine harmonic d) Odd sine harmonic

6) Expand in complex Fourier

a) $f(x) = e^x$, $-1 < x < 1$, b) $f(x) = x$, $-3 < x < 3$

7) Expand the function $f(x) = 1$, $0 < x < T$ in

- (a) Fourier cosine (b) Fourier sine (c) Odd harmonic
- (d) Even harmonic (e) Even Cosine harmonic
- (f) Even Sine harmonic

8) Find Fourier transforms of the following functions

a) $f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$, then evaluate $\int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos(x/2) dx$

$$b) f(x) = \begin{cases} 1 & 0 < x < a \\ -1 & -a < x < 0 \\ 0 & |x| > a \end{cases}$$

9) Expand the following functions in Fourier sine series:

$$a) f(x) = \begin{cases} \pi/2 - x, & 0 \leq x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi \end{cases},$$

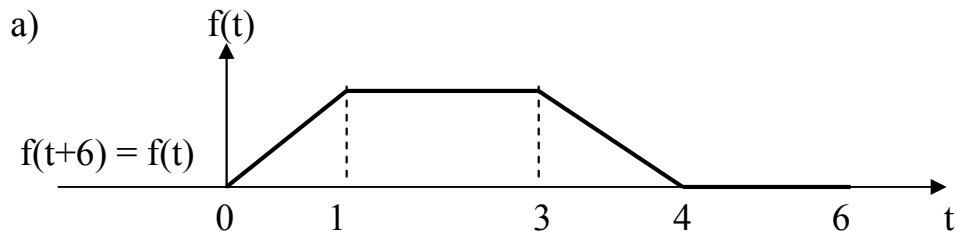
$$b) f(x) = e^x, \quad 0 < x < 1$$

10) If $f(x) = \begin{cases} 0 & x \leq -\pi/2 \\ \pi/2 + x & -\pi/2 < x \end{cases}$, then the first Fourier cosine coefficient equaland the second Fourier sine coefficient equal

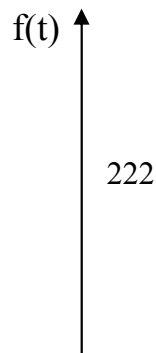
11) Find the first Fourier cosine coefficients of the functions:

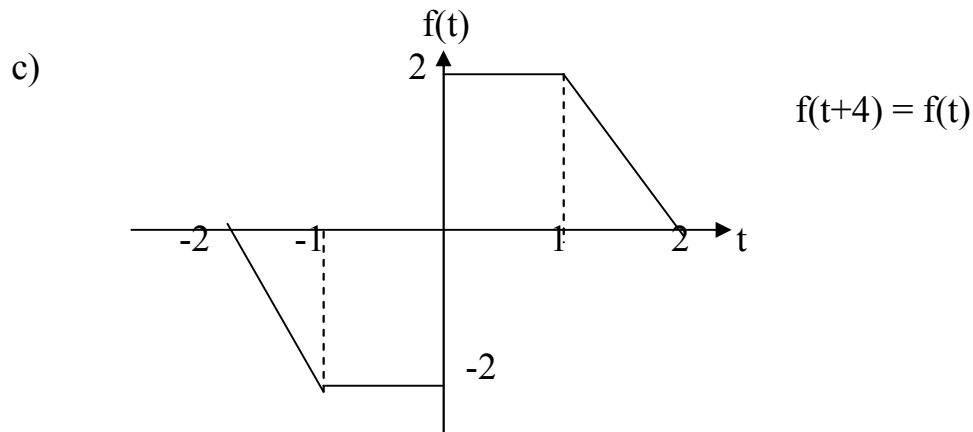
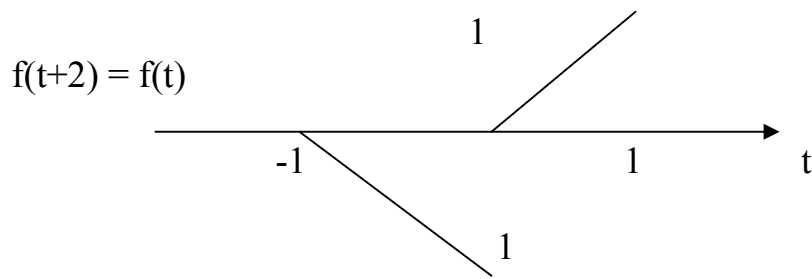
$$a) f(x) = \begin{cases} x-5 & 0 < x \leq 5 \\ 0 & 5 < x \leq 10 \end{cases}, \quad b) f(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1-x & 1 < x \leq 2 \end{cases},$$

12) Expand the following functions in Fourier series:



b)





13) Suppose $f(x) = \pi x^2 - 2x^3$ on $[0, \pi]$ and $g(x)$ is the sum of the whole Fourier sine series for $f(x)$ and $h(x)$ is the sum of the whole Fourier cosine series for $f(x)$, compute $g(1)$, $h(1)$, $g(\pi)$, $h(\pi)$.

14) Prove: $x = \frac{4}{\pi} \left[\sin x - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} - \dots \right]$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

15) Find the Fourier series of the function f defined by

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases},$$

16) Find the Fourier series of the function f defined by

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 3, & 0 < x < \pi \end{cases},$$

17) Let h be a given number in the interval $(0, \pi)$. Find the Fourier

cosine series of the function $f(x) = \begin{cases} 1, & 0 < x < h \\ 0, & h < x < \pi \end{cases},$

18) Calculate the Fourier sine series of the function defined by

$f(x) = x(\pi - x)$ on $(0, \pi)$. Use its Fourier representation to find the

value of the infinite series $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \dots$

19) Let h be a given number in the interval $(0, \pi)$. Find the Fourier

cosine series representation of $f(x) = \begin{cases} 1 & 0 < x < h \\ \frac{2h-x}{2h} & h < x < 2h \\ 0 & 2h < x < \pi \end{cases}$

20) What is the Fourier sine series and Fourier cosine series of $f(x) = \pi/4 - x/2$, where $0 < x < \pi$.

21) Find the Fourier series of $f(x) = |x|$ where $-L < x < L$.

22) The function f is defined by $f(x) = e^x$ for $-L < x < L$. Find its Fourier series.

23) Let a be a given integer. The function f is defined by $f(x) = \sin ax$ for $0 < x < \pi$. Find its Fourier cosine series.

24) Let f be a periodic function of period 2π such that $f(x) = \pi^2 - x^2$ for $x \in (-\pi, \pi)$. show that $\pi^2/12 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

25) Find the Fourier series of $f(x) = 3x$ for $x \in (-\pi, \pi)$, then deduce the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

26) Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases}$$
, sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$. Find sum of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ & $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$.

27) Find the Fourier series of $f(x) = x/2$ for $x \in (-\pi, \pi)$, then deduce the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

28) Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$
, Sketch a graph of $f(x)$ in the interval $-2\pi < x < 2\pi$ and find the sum of $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$.

29) Find the Fourier series of $f(x) = x^2$ for $x \in (-\pi, \pi)$, then deduce the sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$.

30) Let the 2π periodic function $f(x)$ be specified on the interval $(-\pi, \pi)$ by $f(x) = x^2$.

(a) Determine the coefficients $\{c_k\}$ and $\{d_k\}$ in the Fourier Series expansion $f(x) = \sum_{k=1}^{\infty} c_k \cos kx + \sum_{k=1}^{\infty} d_k \sin kx$, then determine the value of $\sum_{k=1}^{\infty} \frac{1}{k^4}$.

31) Determine the Fourier series to $f(x) = e^{-|x|}$, $g(x) = \sin(3x/2)$, $h(x) = x \sin(x)$, $-\pi < x < \pi$.

32) $f(x) = e^{-4x}$ for $-2 < x < 2$ with $f(x+4) = f(x)$.

33) $f(t) = \begin{cases} \pi^2, & -\pi < t < 0 \\ (t-\pi)^2, & 0 < t < \pi \end{cases}$, with $f(t) = f(t+2\pi)$. Show how to

use this Fourier series to compute the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$

34) Expand in Fourier series

a) $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi/2 & \pi/2 < x < \pi \\ \pi - x/2 & \pi < x < 2\pi \end{cases}$, with $f(x) = f(x+2\pi)$

b) $f(x) = \begin{cases} \sin(t/2), & 0 < t < \pi \\ -\sin(t/2), & \pi < t < 2\pi \end{cases}$, with $f(t) = f(t+2\pi)$

Probability problems

- (1) Find the expectation of the sum of points in tossing a pair of fair dice.
- (2) Find the expectation of a discrete random variable X whose probability function is given by $f(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \dots$
- (3) A continuous random variable X has probability density given by $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

Find $E(X), E(X^2)$

- (4) The joint density function of two random variables X and Y is given by $f(x, y) = \begin{cases} xy/96 & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$

Find $E(X), E(Y), E(XY), E(2X + 3Y)$

- (5) Find (a) the variance and (b) the standard deviation of the sum obtained in tossing a pair of fair dice.
- (6) Find (a) the variance and (b) the standard deviation for a continuous random variable X which has probability density given by $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

- (7) Prove that the quantity $E[(X - a)^2]$ is minimum when $a = \mu = E(X)$.

- (8) Prove that $E[(X - \mu)^2] = E(X) - [E(X)]^2$.

- (9) Prove that $Var(X + Y) = Var(X) + Var(Y)$.

- (10) If X and Y are two continuous random variables having joint density function $f(x, y)$ show that $\sigma_{XY} = E(XY) - E(X).E(Y)$.
- (11) The joint probability function of two discrete random variables X and Y is given by $f(x, y) = c(2x + y)$ where x and y can assume all integers such that $0 \leq x \leq 2$, $0 \leq y \leq 3$ and $f(x, y) = 0$ otherwise. considering $c = 1/42$ Find,
 (a) $E(X)$, (b) $E(Y)$, (c) $E(XY)$, (d) $E(X^2)$,
 (e) $E(Y^2)$, (f) $Var(X)$, (g) $Var(Y)$, (h) $Cov(X, Y)$, (i) ρ
- (12) The joint probability function of two discrete random variables X and Y is given by

$$f(x, y) = \begin{cases} c(2x + y) & 2 \leq x \leq 6, \quad 0 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$
 . Find,
 (a) $E(X)$, (b) $E(Y)$, (c) $E(XY)$, (d) $E(X^2)$,
 (e) $E(Y^2)$, (f) $Var(X)$, (g) $Var(Y)$, (h) $Cov(X, Y)$, (i) ρ



- Answer all the following questions
- No. of questions:4
- Total Mark: 80 Marks

1-a) Find Fourier series for the function $f(x) = |\sin x|$ $-\pi \leq x \leq \pi$, and

$$\text{find } \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2(2m+1)^2}.$$

1-b) Expand into complex Fourier series the periodic function $f(x) =$

$$\begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \text{ of period } 2\pi \quad \textbf{(15 marks)}$$

2-a) Find generating function for Legendre polynomial and show that:

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n-1} P_{n-1}(x), \text{ then find } P_2(x), P_3(x).$$

2-b) Write the recurrence relation for Bessel's function and show that

$$\int x^n J_{n-1}(x) dx = x^n J_n(x) + c, \text{ find } J_{5/2}(x) \quad \textbf{(20 marks)}$$

Probability and Statistics (Total scores :45marks)

3a-i) Susan goes to work by one of two routes A or B. The prob. of going by route A is 30%. If she goes by route A, the prob. of being late is 5% and if she goes by route B, the prob. of being late is 10%. Given Susan is late for school, find prob. that she went via route B. **(5 marks)**

3a-ii) Suppose that you have a bag filled with 50 marbles, 15 of which are green. What is the prob. of choosing exactly 3 green marbles if a total of 10 marbles are selected? **(5 marks)**

3b-i) Let the r.v. X be the distance in feet between bad records on a used computer tape. Suppose that a reasonable probability model for X is given by the p.d.f.

$f(x) = \frac{1}{40}e^{-x/40}$, $0 < x < \infty$, find $P(X > x)$ and then the median, also find m.g.f.

and then deduce mean, standard deviation, μ_3, μ_3' . **(10 marks)**

3b-ii) Suppose that (X, Y) has probability density function f given by $f(x, y) = (x + y) / 4$ for $0 < x < y < 2$

Find the probability density functions of X and of Y (Marginal of X and Y), $P[(X+Y) > 1/2]$ and then determine if X and Y are independent. **(10 marks)**

4-a) A fair coin is tossed three times, X is the N^o of heads that come up on the first 2 tosses and Y is the N^o of heads that come up on tosses 2, 3. Construct the joint distribution and find marginal of X and Y , also find expected value and variance of X and Y **(10 marks)**

4-b) Suppose that if a car gets into an accident, the dollar amount of damage is Gamma-distributed with $\alpha = 2$ and $\beta = 100$. Evaluate the average loss (in dollars) and the standard deviation of the loss (to the nearest dollar)?

(5 marks)

BOARD OF EXAMINERS Dr. Ibrahim Sakr & Dr. Khaled El Naggar

Model Answer

Probability and Statistics

3a-i) $P(A) = 0.3$, $P(B) = 0.7$, $P(L/A) = 0.05$, $P(L/B) = 0.1$, $P(B/L) = [P(L/B)P(B)]/P(L)$, where L: is Late event, $P(L) = P(L/A)P(A) + P(L/B)P(B) = 0.05(0.3) + 0.1(0.7) = 0.085$, so $P(B/L) = 0.1(0.7)/0.085 = 0.824$

3a-ii) Let the r.v. is the number of green marbles, so that by using hypergeometric distribution $N = 50$, $k = 15$, $n = 10$, therefore $P(X = 3) = \frac{{}^{50}C_3 \cdot {}^{35}C_7}{{}^{50}C_{10}}$.

3b-i) $P(X > x) = 1 - P(X \leq x) = 1 - \int_0^x \frac{1}{40} e^{-x/40} dx = e^{-x/40}$. To get the median a such

that $P(X < a) = 0.5$, therefore $\int_0^a \frac{1}{40} e^{-x/40} dx = 0.5$, thus $1 - e^{-a/40} = 0.5 \Rightarrow a = -40$

$\ln(0.5) = 27.726$, & m.g.f. $= \int_0^\infty e^{tx} \left(\frac{1}{40} e^{-x/40} \right) dx = \int_0^\infty \frac{1}{40} e^{\frac{-(1-40t)x}{40}} dx = \frac{1}{1-40t} = \phi(t)$,

thus $\mu_1' = \phi'(0) = \frac{40}{(1-40t)^2} \Big|_{t=0} = 40 = E(X)$ and $\mu_2' = \phi''(0) = \frac{3200}{(1-40t)^3} \Big|_{t=0} =$

$3200 = E(X^2)$, hence $\text{var}(X) = E(X^2) - [E(X)]^2 = 3200 - 1600 = 1600$, so standard

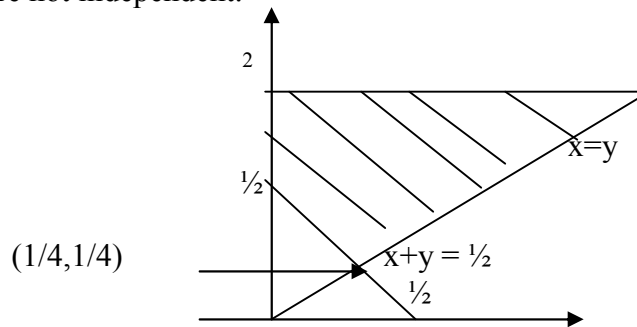
deviation $= 40$ and $\mu_3' = \phi'''(0) = \frac{384000}{(1-40t)^3} \Big|_{t=0} = 384000$, but $\mu_3 = \mu_3' -$

$3\mu_1' \mu_2' + 2[\mu_1']^3$, therefore $\mu_3 = 384000 - 3(40)(3200) + 2(40)^3 = 128000$.

3b-ii) $f_1(x) = \int_x^2 \frac{(x+y)}{2} dy = \frac{xy + y^2/2}{2} \Big|_x^2 = x + 1 - \frac{3x^2}{4}$ and

$f_2(y) = \int_0^y \frac{(x+y)}{2} dx = \frac{xy + x^2/2}{2} \Big|_0^y = \frac{3y^2}{4}$

$$P[(X+Y) > 1/2] = \int_{1/4}^{1/2} \left[\int_{1/2-y}^y \left(\frac{x+y}{2} \right) dx \right] dy + \int_{1/2}^2 \left[\int_0^y \left(\frac{x+y}{2} \right) dx \right] dy$$
 and since $f_1(x) \neq f_2(y)$, thus they are not independent.



4-a)

X \ Y	0	1	2	$f_1(x)$
0	1/8	1/8	0	2/8
1	1/8	2/8	1/8	4/8
2	0	1/8	1/8	2/8
$f_2(y)$	2/8	4/8	2/8	1

$E(X) = 0(2/8) + 1(4/8) + 2(2/8) = 1$, $E(Y) = 0(2/8) + 1(4/8) + 2(2/8) = 1$, $E(X^2) = 1(4/8) + 4(2/8) = 3/2$, $E(Y^2) = 1(4/8) + 4(2/8) = 3/2$, $\text{Var}(X) = \text{Var}(Y) = 1/2$

4b) Since $E(X) = \alpha / \beta = 2/100 = 0.02$, $\text{Var}(X) = \alpha / \beta^2$, therefore standard deviation $= \sqrt{2} / 100$



- Answer all the following question
- Illustrate your answers with sketches when necessary.
- The exam. Consists of one page
- No. of questions:4
- Total Mark: 80 Marks

1-a) Find Fourier series for the function $f(x) = x$, $0 \leq x \leq \pi/2$, period = 2π in even

cosine harmonic and find $\sum_{m=1}^{\infty} \frac{1}{(2m-1)^4}$.

1-b) Expand into complex Fourier series the periodic function $f(x) =$

$$\begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \text{ of period } 2\pi \quad \textbf{(25 marks)}$$

2) Solve the linear programming problem:

Max $f = 2x + y + 4z$, Subject to: $x + y + 2z \leq 20$, $2x + 3y + 2z = 18$, $x + 2y + 2z \geq 6$, $x, y, z \geq 0$ **(20 marks)**

Probability and Statistics

3-a) Suppose that there is a test for a certain disease. If a person with the disease takes the test, then the test will come back positive 95% of the time (like most medical tests it isn't foolproof). On the other hand, the test will show that you are positive 3.5% when you do not have the disease.

- (i) Given that I test positive, what is the chance that I have the disease?
- (ii) Given that someone in the sample tests negative, what is the probability that (s)he really does not have the disease?

3-b) A coin is biased so that heads is twice the tails for three independent tosses of the coin, **find** (i) The probability of getting at most two heads.

(ii) C.d.f. of the random variable X, and use it to find $P(1 < X \leq 3)$; $P(X > 2)$.

(15 marks)

4-a) Let X and Y denote the amplitude of noise signals at two antennas. The random vector (X, Y) has the joint pdf $f(x, y) = ax e^{-ax^2/2} by e^{-by^2/2}$, $x > 0, y > 0$, $a > 0, b > 0$, **find** (i) $P[X > Y]$ (ii) Standard deviation of X

4b-i) Derive m.g.f. for gamma distribution, then deduce μ_r' , $r = 0, 1, 2, 3$

4b-ii) Given a bag containing 3 black balls, 2 blue balls and 3 green balls, a random sample of 2 balls is selected. Given that X is the number of black balls and Y is the number of blue balls, find the joint probability distribution of X and Y and $Cov(X, Y)$.

(20 marks)

Good luck

BOARD OF EXAMINERS: **Dr. Ibrahim Sakr & Dr.**

Khaled El Naggar

Model answer

3a) Let disease: D, and doesn't have disease: D' , P: positive, N: negative such that $P(D) = P(D') = 0.5$ and $P(P/D) = 0.95$, therefore $P(N/D) = 0.05$, $P(P/D') = 0.035$, thus $P(N/D') = 0.965$, so $P(D/P) = [P(P/D)P(D)]/P(P) = (0.95)(0.5)/[0.95(0.5) + 0.035(0.5)]$ and $P(D'/N) = P(N/D') P(D')/P(N) = 0.965(0.5)/[0.05(0.5) + 0.965(0.5)]$

3b-i) $P(H) = 2 P(T)$, therefore $P(H) = 2/3 = P$, and $P(H \leq 2) =$

$$\sum_{x=0}^2 {}^3C_x (2/3)^x (1/3)^{3-x}$$

ii) $F(x=0) = {}^3C_0 (2/3)^0 (1/3)^3$, $F(x=1) = \sum_{x=0}^1 {}^3C_x (2/3)^x (1/3)^{3-x}$, $F(x=2) =$

$$\sum_{x=0}^2 {}^3C_x (2/3)^x (1/3)^{3-x},$$

$$F(x=3) = \sum_{x=0}^3 {}^3C_x (2/3)^x (1/3)^{3-x}, \quad P(1 < X \leq 3) = F(x=3) - F(x=1) =$$

$$\sum_{x=0}^3 {}^3C_x (2/3)^x (1/3)^{3-x} - \sum_{x=0}^1 {}^3C_x (2/3)^x (1/3)^{3-x}, \quad P(X > 2) = F(x=3) - F(x=2) =$$

$$\sum_{x=0}^3 {}^3C_x (2/3)^x (1/3)^{3-x} - \sum_{x=0}^2 {}^3C_x (2/3)^x (1/3)^{3-x}$$

4a-i) $P(X > Y)$

$$= \int_0^\infty \int_y^\infty axe^{-ax^2/2} bye^{-by^2/2} dx dy = \int_0^\infty -e^{-ax^2/2} \Big|_y^\infty bye^{-by^2/2} dy = \int_0^\infty e^{-ay^2/2} bye^{-by^2/2} dy$$

$$= \int_0^\infty bye^{-(a+b)y^2/2} dy = -\frac{b}{a+b} e^{-(a+b)y^2/2} \Big|_0^\infty = \frac{b}{a+b}, \quad f_1(x) =$$

$$\int_0^\infty [axe^{-ax^2/2} bye^{-by^2/2}] dy,$$

$$E(X) = \int_0^\infty xf_1(x) dx \quad \text{and} \quad E(X^2) = \int_0^\infty x^2 f_1(x) dx, \quad \text{therefore} \quad \text{Var}(X) = E(X^2) - [E(X)]^2$$

4b-i) The moment generating function can be expressed by

$$E(e^{tx}) = \int_0^\infty e^{tx} \left(\frac{\beta^\alpha}{\Gamma \alpha} x^{\alpha-1} e^{-\beta x} \right) dx = \frac{\beta^\alpha}{\Gamma \alpha} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx$$

$$\text{Put } (\beta-t)x = y \Rightarrow dx = \frac{dy}{\beta-t}, \quad \text{thus } E(e^{tx}) = \frac{\beta^\alpha}{(\beta-t)^\alpha \Gamma \alpha} \int_0^\infty y^{\alpha-1} e^{-y} dy = \frac{\beta^\alpha}{(\beta-t)^\alpha},$$

$$\mu_0' = 1, \quad \mu_1' = E(X) = \alpha / \beta, \quad \mu_2' = E(X^2) \quad \text{and} \quad \mu_3' = \frac{d^3}{dt^3} \left[\frac{\beta^\alpha}{(\beta-t)^\alpha} \right] \Big|_{t=0}$$

4b-ii) B: Black, b: Blue, G: green

X \ Y	0	1	2	$f_1(x)$
0	$P(GG) = 0.1071$	$2P(BG) = 0.3214$	$P(BB) = 0.1071$	0.5356
1	$2P(bG) = 0.2143$	$2P(Bb) = 0.2143$	0	0.4286
2	$P(bb) = 0.0357$	0	0	0.0357
$f_1(x)$	0.3571	0.5357	0.1071	1

$E(Y) = 0(0.5356) + 1(0.4286) + 2(0.0357) = 0.5$, $E(X) = 0(0.3571) + 1(0.5357) + 2(0.1071) = 0.75$, $E(XY) = 1(0.2143) = 0.2143$, therefore
 $Cov(X,Y) = E(XY) - E(X)E(Y) = -0.1607$

Benha University
Faculty of Engineering- Shoubra
Electrical Engineering Department
2nd Year electrical power



Final Term Exam
Date: 1st of January 2012
Mathematics 3a
Duration : 3 hours

- Answer all the following questions
- No. of questions:4
- Total Mark: 80 Marks

1-a) A box contains 7 blue , 8 white and 9 red balls, two balls are drawn without replacement. Let X is the number of blue balls and Y is the number of red balls, Find joint probability function, $Cov(x,y)$, $P(X+Y=2)$. **(12 marks)**

1-b) Evaluate m.g.f. for the random variable of exponential distribution, then deduce μ'_r , $r = 0,1,2$ **(8 marks)**

2-a) $f(x,y) = \begin{cases} cxy & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$. Find $Cov(x,y)$, $P[(X+Y)<1/2]$

(12 marks)

2-b) A urn contains 30 red balls and 20 black balls, sample of 5 balls is selected at random. Let X is the number of red balls, find $P(X=3)$, $E(X)$, $Var(X)$.

(8 marks)

3-a) Expand in fourier series the following periodic functions:

$$f(x) = x|x|, -1 < x < 1, \quad f(x) = x^2, \quad 0 < x < 2, \text{ then deduce } \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \textbf{(12 marks)}$$

3-b) Find Fourier transform and Fourier integral $f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$ **(8 marks)**

4-a) Solve the L.P.P. that maximize $z = 2x - 3y$, s.t. $x - 2y < 3$, $2x + y < 5$, $x, y > 0$ using two different methods, and find the dual formula. **(12 marks)**

4-b) Solve the L.P.P. that minimize $z = x - 3y$, s.t. $x - 2y > 3$, $2x + y > 5$, $x, y > 0$ using Simplex method. **(8 marks)**

BOARD OF EXAMINER

Dr. Khaled El Naggar

Model answer

1-a)

X \ Y	0	1	2	$f_1(x)$
0	$P(WW) = 0.1014$	$2P(BW) = 0.2029$	$P(BB) = 0.0761$	0.3804
1	$2P(RW) = 0.2609$	$2P(BR) = 0.2283$	0	0.4892
2	$P(RR) = 0.1304$	0	0	0.1304
$f_1(x)$	0.4927	0.4312	0.0761	1

$$P(X+Y=2) = f(1,1) + f(2,0) + f(0,2) = 0.2283 + 0.0761 + 0.1304 = 0.4348$$

$$E(Y) = 0(0.3804) + 1(0.4892) + 2(0.1304) = 0.75, \quad E(X) = 0(0.4927) + 1(0.4312) + 2(0.0761) = 0.5834, \quad E(XY) = 0(0.1014) + 0(0.2029) + 0(0.0761) + 0(0.2609) + 1(0.2283) + 2(0) + 0(0.1304) + 2(0) + 4(0) = 0.2283, \\ \text{therefore } \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = -0.2093$$

1-b) The moment generating function of a exponential distribution is expressed by

$$E(e^{tx}) = \int_0^{\infty} e^{tx} (\lambda e^{-\lambda x}) dx = \int_0^{\infty} \lambda e^{-(\lambda-t)x} dx = \frac{\lambda}{(\lambda-t)},$$

$$\mu'_0 = 1, \mu'_1 = E(X) = \frac{1}{\lambda}, \mu'_2 = E(X^2) = \frac{2}{\lambda^2}$$

2-a) First we have to get c, such that

$$\int_{y=0}^1 \int_{x=y}^1 cxy \, dx \, dy = 1 \Rightarrow \int_{y=0}^1 \frac{y-y^3}{2} \, dy = 1 \Rightarrow c = 8$$

The marginal probabilities $f_1(x)$, $f_2(y)$ are expressed by:

$$f_1(x) = \int_0^x 8xy \, dy = 4xy^2 \Big|_0^x = 4x^3 \text{ and } f_2(y) = \int_y^1 8xy \, dx = 4yx^2 \Big|_y^1 = 4(y - y^3)$$

$$E(X) = \int_0^1 xf_1(x) \, dx = \int_0^1 4x^4 \, dx = 4/5, E(Y) =$$

$$\int_0^1 yf_2(y) \, dy = \int_0^1 4y(y - y^3) \, dy = 8/15$$

$$E(XY) = \int_{y=0}^1 \int_{x=y}^1 8x^2y^2 \, dx \, dy = 4/9, \text{ Cov}(X,Y) = E(XY) - E(X)E(Y) = 4/9 - (4/5)(8/15) = 0.0177$$

$$P(X+Y < 1/2) = \int_{y=0}^{1/4} \int_{x=y}^{1/2-y} 8xy \, dx \, dy = \int_{y=0}^{1/4} 4x^2y \Big|_{x=y}^{1/2-y} \, dy = \int_{y=0}^{1/4} (1-4y)y \, dy = 5/6$$

2-b) $N = 50$, $k = 30$, $n = 5$, therefore by using hypergeometric distribution

$$P(X=3) = [{}^{30}C_3] [{}^{20}C_2] / [{}^{50}C_5]$$

$$E(X) = n(k/N) = 5(30/50) \text{ and } V(X) = \left(\frac{N-n}{N-1} \right) n \left(\frac{k}{N} \right) \left(1 - \frac{k}{N} \right) = 5 \left(\frac{45}{49} \right) \left(\frac{30}{50} \right) \left(\frac{20}{50} \right)$$

3-a) The functions $f(x) = x|x|$ is even, therefore $a_0 = a_n = 0$, $T = 1$, therefore

$$b_n = \frac{2}{1} \int_0^1 x^2 \sin\left(\frac{n\pi x}{1}\right) dx =$$

$$\left[x^2 \left(-\frac{1}{n\pi} \cos\left(\frac{n\pi x}{1}\right) \right) - 2x \left(-\frac{1}{n^2\pi^2} \sin\left(\frac{n\pi x}{1}\right) \right) + 2 \left(\frac{1}{n^3\pi^3} \cos\left(\frac{n\pi x}{1}\right) \right) \right]_0^1$$

$$= \left[\left(-\frac{1}{n\pi} \cos\left(\frac{n\pi}{1}\right) \right) + 2 \left(\frac{1}{n^3\pi^3} \cos\left(\frac{n\pi}{1}\right) - \frac{1}{n^3\pi^3} \right) \right], \text{ thus } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right) =$$

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

This function is $f(x) = x^2$ neither even nor odd, therefore

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right), \quad T=1$$

$$a_0 = \frac{1}{1} \int_0^2 f(x) dx = \frac{1}{1} \int_0^2 (x^2) dx = \left(\frac{x^3}{3} \right) \Big|_0^2 = \frac{8}{3}$$

$$a_n = \frac{1}{1} \int_0^2 (x^2) \cos\left(\frac{n\pi x}{1}\right) dx = \left[(x^2) \left(\frac{1}{n\pi} \sin\left(\frac{n\pi x}{1}\right) \right) - (2x) \left(-\frac{1}{n^2\pi^2} \cos\left(\frac{n\pi x}{1}\right) \right) \right.$$

$$\left. + 2 \left(-\frac{1}{n^3\pi^3} \sin\left(\frac{n\pi x}{1}\right) \right) \right]_0^2 = \frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{1} \int_0^2 (x^2) \sin\left(\frac{n\pi x}{1}\right) dx = \left[(x^2) \left(-\frac{1}{n\pi} \cos\left(\frac{n\pi x}{1}\right) \right) - (2x) \left(-\frac{1}{n^2\pi^2} \sin\left(\frac{n\pi x}{1}\right) \right) \right.$$

$$\left. + 2 \left(\frac{1}{n^3\pi^3} \cos\left(\frac{n\pi x}{1}\right) \right) \right]_0^2 = \frac{-4}{n\pi}$$

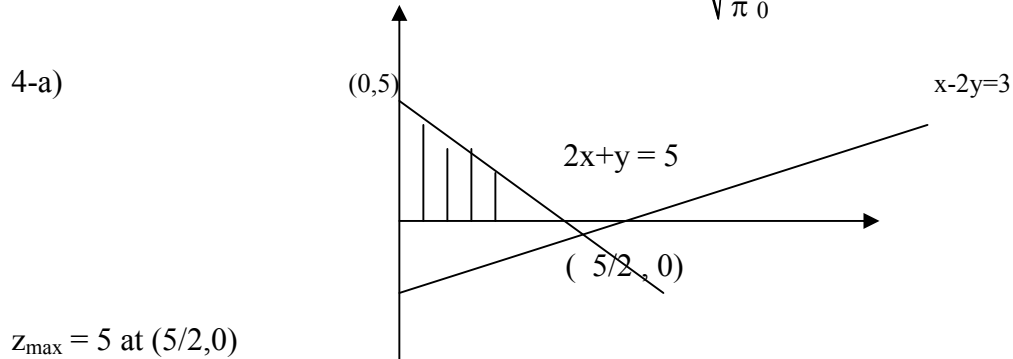
3-b) Since this function is even, therefore there is only Fourier Cosine transform

$$\text{such that } F_c(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x dx = \sqrt{2/\pi} \int_0^1 (1-x^2) \cos \alpha x dx$$

$$= \sqrt{2/\pi} \left[(1-x^2) \left(\frac{\sin \alpha x}{\alpha} \right) - (-2x) \left(\frac{-\cos \alpha x}{\alpha^2} \right) + (-2) \left(\frac{-\sin \alpha x}{\alpha^3} \right) \right]_0^1$$

$$= \sqrt{2/\pi} \left[\frac{-2\cos \alpha}{\alpha^2} + \frac{2\sin \alpha}{\alpha^3} \right] = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin \alpha - \alpha \cos \alpha}{\alpha^3} \right]$$

Therefore Fourier integral $f(x)$ is expressed by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha x \, d\alpha$



$z_{\max} = 5$ at $(5/2, 0)$

Dual problem is $L_{\min} = 3p + 5q$ s.t. $p+2q > 2$, $-2p+q > -3$

4-b) We can solve it using dual problem such that $L_{\max} = 3p + 5q$, s.t. $p + 2q < 1$, $-2p + q < -3$,

therefore $2p - q > 3$ so that $p+2q + s = 1$, $2p - q - t + u = 3$, where s is slack variable, t is surplus and u is artificial such that $w = u \Rightarrow w-3 = -2p + q + t$

	p	q	s	t	u	Const.
s	1	2	1	0	0	1
u	2	-1	0	-1	1	3
-L	3	5	0	0	0	0
-w	-2	1	0	1	0	-3

	p	q	s	t	u	Const.
p	1	2	1	0	0	1
u	0	-5	-2	-1	1	1
-L	0	-1	-3	0	0	-3
-w	0	5	2	1	0	-1

Since the row of w has only positive elements or zeros, therefore there is no solution.



- Answer all the following question
- Illustrate your answers with sketches when necessary.
- The exam. Consists of one page
- No. of questions:4
- Total Mark: 80 Marks

- 1-a) Suppose there are 5 black, 10 white, and 15 red marbles in an urn. You reach in and randomly select six marbles without replacement. What is the probability that you pick exactly two red marbles? Find $E(X)$ and $\text{var}(X)$ **(10marks)**
- 1-b) Let X be a random variable with gamma distribution with $\alpha = 2$, $\beta = 1/5$. Find the probability $P(X > 30)$, $E(X)$ and $\text{Var}(X)$ **(10 marks)**
- 2-a) A box contains 7 blue , 8 white and 9 red balls, two balls are drawn without replacement. Let X is the number of blue balls and Y is the number of red balls, Find joint probability function, $\text{Cov}(x,y)$, $P(X+Y=2)$. **(10 marks)**

2-b) $f(x,y) = \begin{cases} cxy & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$ Find $\text{Cov}(x,y)$, $P[(X+Y)<1/2]$ **(10 marks)**

- 3-a) Expand in fourier series $f(x) = |\sin x|$, $0 < x < 2\pi$, then deduce

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$
 (12 marks)

- 3-b) Expand into complex Fourier series the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \text{ of period } 2\pi$$
 (8 marks)

4-a) Solve the integral equation $\int_0^{\infty} f(x) \sin(\alpha x) dx = \begin{cases} 1 & 0 \leq \alpha < 1 \\ 2 & 1 \leq \alpha < 2 \\ 0 & 2 \leq \alpha \end{cases}$ **(10 marks)**

- 4-b) $f(x) = x$, $0 < x < 1$, expand in (i) cosine harmonic (ii) odd harmonic **(10marks)**

BOARD OF EXAMINERS

Dr. eng. Khaled El Naggar

Model answer

1-a) $n = 6$, $N = 30$, $k = 15$, therefore $P(X = 2) = [{}^{15}C_2] [{}^{15}C_4] / [{}^{30}C_6]$ and

$$E(X) = n(k/N) = 6(15/30) = 3, \text{ and } V(X) = \left(\frac{N-n}{N-1}\right)n\left(\frac{k}{N}\right)\left(1-\frac{k}{N}\right) = \frac{36}{29}$$

$$1\text{-b) } P(X > 30) = \frac{1}{25} \int_{30}^{\infty} x e^{-x/5} dx, \text{ put } y = x-30, \text{ therefore}$$

$$P(X > 30) = \frac{1}{25} \int_0^{\infty} (y+30) e^{-(y+30)/5} dy = \frac{e^{-6}}{25} \int_0^{\infty} y e^{-y/5} dy + \frac{6e^{-6}}{5} \int_0^{\infty} e^{-y/5} dy$$

Put $y/5 = z \Rightarrow dz = dy/5$, therefore

$$P(X > 30) = e^{-6} \int_0^{\infty} z e^{-z} dz + 6e^{-6} \int_0^{\infty} e^{-z} dz = 7e^{-6}$$

$$E(X) = \alpha / \beta = 10, \text{ Var}(X) = 50$$

2-a)

X \ Y	0	1	2	$f_1(x)$
0	$P(WW) = 0.1014$	$2P(BW) = 0.2029$	$P(BB) = 0.0761$	0.3804
1	$2P(RW) = 0.2609$	$2P(BR) = 0.2283$	0	0.4892
2	$P(RR) = 0.1304$	0	0	0.1304
$f_1(x)$	0.4927	0.4312	0.0761	1

$$P(X+Y=2) = f(1,1) + f(2,0) + f(0,2) = 0.2283 + 0.0761 + 0.1304 = 0.4348$$

$$E(Y) = 0(0.3804) + 1(0.4892) + 2(0.1304) = 0.75, \quad E(X) = 0(0.4927) + 1(0.4312) + 2(0.0761) = 0.5834, \\ E(XY) = 0(0.1014) + 0(0.2029) + 0(0.0761) + 0(0.2609) + 1(0.2283) + 2(0) + 0(0.1304) + 2(0) + 4(0) = 0.2283, \\ \text{therefore } \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = -0.2093$$

2-b) P First we have to get c, such that

$$\int_{y=0}^1 \int_{x=y}^1 cxy \, dx \, dy = 1 \Rightarrow \int_{y=0}^1 \frac{y-y^3}{2} \, dy = 1 \Rightarrow c = 8$$

The marginal probabilities $f_1(x)$, $f_2(y)$ are expressed by:

$$f_1(x) = \int_0^x 8xy \, dy = 4xy^2 \Big|_0^x = 4x^3 \text{ and } f_2(y) = \int_y^1 8xy \, dx = 4yx^2 \Big|_y^1 = 4(y - y^3)$$

$$E(X) = \int_0^1 xf_1(x) \, dx = \int_0^1 4x^4 \, dx = 4/5, \quad E(Y) = \int_0^1 yf_2(y) \, dy = \int_0^1 4y(y - y^3) \, dy = 8/15$$

$$E(XY) = \int_{y=0}^1 \int_{x=y}^1 8x^2y^2 \, dx \, dy = 4/9, \quad \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 4/9 - (4/5)(8/15) = 0.0177$$

$$P(X+Y < 1/2) = \int_{y=0}^{1/4} \int_{x=y}^{1/2-y} 8xy \, dx \, dy = \int_{y=0}^{1/4} 4x^2y \Big|_{x=y}^{1/2-y} \, dy = \int_{y=0}^{1/4} (1-4y)y \, dy = 5/6$$

3-a) This function is even cosine harmonic, therefore $a_0 = \frac{4}{T} \int_0^{T/2} f(x) \, dx =$

$$\frac{4}{\pi} \int_0^{\pi/2} \sin(x) \, dx = \frac{4}{\pi}$$

$$a_{2n} = \frac{4}{T} \int_0^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) \, dx = \frac{4}{\pi} \int_0^{\pi/2} \sin(x) \cos(2nx) \, dx = -\frac{2}{\pi(2n-1)(2n+1)},$$

$$b_{2n} = 0$$

$$\text{Thus } |\sin x| = \frac{a_0}{2} \sum_{n=1}^{\infty} a_{2n} \cos(2nx) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos(2nx), \text{ put } x = 0,$$

$$\text{therefore } \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = 1$$

3-b) Since $T = \pi$, therefore

$$c_n = \frac{1}{2\pi} \int_{-T}^T f(x) e^{-i\left(\frac{n\pi x}{T}\right)} dx = \frac{1}{2\pi} \int_0^\pi e^{-i(nx)} dx = \frac{i}{2\pi n} [e^{-i(n\pi)} - 1] = \frac{i}{2\pi n} [\cos(n\pi) - 1]$$

$$\text{Thus } c_{2n-1} = \frac{-i}{\pi n}, \text{ therefore } f(x) = \sum_{n=-\infty}^{\infty} c_{2n-1} e^{-i(2n-1)x},$$

$$4\text{-a) } F_S(\alpha) = \sqrt{2/\pi} \int_0^\infty \sin \alpha x dx = \sqrt{2/\pi} \begin{cases} 1 & 0 \leq \alpha < 1 \\ 2 & 1 \leq \alpha < 2 \\ 0 & 2 \leq \alpha \end{cases}, \text{ therefore}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(\alpha) \sin \alpha x d\alpha = \frac{2}{\pi} \left[\int_0^1 \sin \alpha x d\alpha + \int_1^2 2 \sin \alpha x d\alpha \right] = \frac{2}{\pi} \left[\frac{1 + \cos \alpha - 2 \cos 2\alpha}{x} \right]$$

4b- i) we have to extend this function to be even such that:

$$a_0 = \frac{2}{1} \int_0^1 x dx = \left(\frac{2x^2}{2} \right)_0^1 = 1$$

$$a_n = \frac{2}{1} \int_0^1 x \cos\left(\frac{n\pi x}{1}\right) dx = 2 \left[x \frac{\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2 \pi^2} \right]_0^1 = 2 \left[\frac{\cos(n\pi) - 1}{n^2 \pi^2} \right]$$

$$\text{Therefore } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2 \pi^2} \cos(2n\pi x),$$

$$4\text{b-ii) Thus } f(x) = \sum_{n=1}^{\infty} a_{2n-1} \cos(2n-1)x + \sum_{n=1}^{\infty} b_{2n-1} \sin(2n-1)x$$

$$\begin{aligned} a_{2n-1} &= \frac{2}{1} \int_0^1 x \cos(2n-1)\pi x dx \\ &= \frac{2}{1} \left(x \left(\frac{\sin(2n-1)\pi x}{(2n-1)\pi} \right) - \left(\frac{-\cos(2n-1)\pi x}{(2n-1)^2 \pi^2} \right) \right)_0^1 = \frac{-4}{(2n-1)^2 \pi^2} \end{aligned}$$

$$b_{2n-1} = \frac{2}{1} \int_0^1 x \sin(2n-1)\pi x \, dx$$

$$= \frac{2}{1} \left(x \frac{-\cos(2n-1)\pi x}{(2n-1)\pi} - \left(\frac{-\sin(2n-1)\pi x}{(2n-1)^2 \pi^2} \right) \right) \Big|_0^1 = \frac{2}{\pi(2n-1)}$$

$$\text{Therefore } f(x) = \sum_{n=1}^{\infty} b_{2n-1} \sin(2n-1)x = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{(2n-1)^2} \sin(2n-1)x$$

Benha University
Faculty of Engineering- Shoubra
Electrical Engineering Department
2nd Year electrical power



Final Term Exam
Date: January 2013
Mathematics 3a
Duration : 3 hours

- Answer all the following question
- Illustrate your answers with sketches when necessary.
- The exam. Consists of one page
- No. of questions:4
- Total Mark: 80 Marks

1a-i) Two fair dice are rolled, what is the probability that the first turns up six, given that the sum is k, for each k from two through 12? **(7 marks)**

1a-ii) A coin is tossed 3 times, let E is the event that heads turn up on first toss, F is the event that tails turn up the second toss. Are E & F independent events? **(5 marks)**

1-b) When Justin is goal-keeper, Shaunie manages to score an average of once for every 10 shots he takes. If Shaunie takes 12 shots, find the following probabilities: He scores (a) exactly twice, (b) at most twice, (c) at least 3 times **(8 marks)**

2-a) The joint density function of two random variables X and Y is given by

$$f(x,y) = \begin{cases} xy/96, & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}, \text{ find } E(X), E(Y), E(XY) \text{ \& Cov}(X,Y)$$

2-b) A fair coin is tossed three times, X is the N^o of heads that come up on the first 2 tosses and Y is the N^o of heads that come up on tosses 2, 3. Construct the joint distribution and find marginal of X and Y, also find expected value and variance of X and Y. **(10 marks)**

3-a) Expand in fourier series the following periodic functions:

i) $f(x) = 10 - x, 5 < x < 15$, ii) $f(x) = |\cos x| \quad 0 < x < 2\pi$ **(12 marks)**

3-b) Find complex Fourier for $f(x) = e^{-x}, -2 < x < 2$ **(8 marks)**

4-a) Solve the integral equation $\int_0^{\infty} f(x) \sin(\alpha x) dx = \begin{cases} 1 & 0 \leq \alpha < 1 \\ 2 & 1 \leq \alpha < 2 \\ 0 & 2 \leq \alpha \end{cases}$ **(10 marks)**

4-b) Solve the follwing L.P,P.that maximize $z = 2x-3y$, s.t. $x-2y < 3, 2x + y < 5, x,y > 0$ using two different methods, and find the dual formula. **(10 marks)**

BOARD OF EXAMINERS

Dr. eng. Khaled El Naggar

Model answer

1a-i) $A = \{ \text{first turns up six} \} = \{ (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \}$

$B = \{ \text{the sum is } k, \text{ for each } k \text{ from two through } 12 \}$, therefore $P(A/B) = 0$ for $k=2,3,\dots,6$.

At $k = 7$, $B = \{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$, thus $P(A/B) = 1/6$

$k = 8$, $B = \{ (2,6), (3,5), (4,4), (5,3), (6,2) \}$, thus $P(A/B) = 1/5$

$k = 9$, $B = \{ (3,6), (4,5), (5,4), (6,3) \}$, thus $P(A/B) = 1/4$

$k = 10$, $B = \{ (4,6), (5,5), (6,4) \}$, thus $P(A/B) = 1/3$

$k = 11$, $B = \{ (5,6), (6,5) \}$, thus $P(A/B) = 1/2$

$k = 12$, $B = \{ (6,6) \}$, thus $P(A/B) = 1$

1a-ii) $P(E) = 1/2$, $P(F) = 1/2$, $P(E \cap F) = 1/4 = P(E)P(F)$, therefore they are independent

1-b) $P(S) = 0.1$, $n = 12$, X is the number of scores is the random variables and $q = 0.9$.

Therefore $P(\text{scores exactly twice}) = P(X=2) = {}^{12}C_2(0.1)^2(0.9)^{10}$ and $P(\text{scores at most twice}) = P(X=0) + P(X=1) + P(X=2) = {}^{12}C_0(0.1)^0(0.9)^{12} + {}^{12}C_1(0.1)^1(0.9)^{11} + {}^{12}C_2(0.1)^2(0.9)^{10}$ and $P(\text{scores at least 3 times}) = \sum_{x=3}^{12} {}^{12}C_x(0.1)^x(0.9)^{12-x}$

2-a) The marginal probabilities $f_1(x)$, $f_2(y)$ are expressed by:

$$f_1(x) = \int_1^5 \frac{xy}{96} dy = \frac{xy^2}{192} \Big|_1^5 = \frac{x}{8} \text{ and } f_2(y) = \int_0^4 \frac{xy}{96} dx = \frac{x^2y}{192} \Big|_0^4 = \frac{y}{12}, \text{ therefore they are independent and } E(X) = \int_0^4 \frac{x^2}{8} dx = \frac{x^3}{24} \Big|_0^4 = \frac{8}{3}, E(Y) = \int_1^5 \frac{y^2}{12} dy = \frac{y^3}{36} \Big|_1^5 = \frac{31}{9}, \text{ but } E(XY) = E(X)E(Y) = \frac{248}{27} \text{ and } E(2X + 3Y) = 2E(X) + 3E(Y) = \frac{16}{3} + \frac{31}{3} = \frac{47}{3}$$

2-b)

X \ Y	0	1	2	$f_1(x)$
0	1/8	1/8	0	2/8
1	1/8	2/8	1/8	4/8
2	0	1/8	1/8	2/8
$f_2(y)$	2/8	4/8	2/8	1

$$E(X) = 0(2/8) + 1(4/8) + 2(2/8) = 1, E(Y) = 0(2/8) + 1(4/8) + 2(2/8) = 1, E(X^2) = 1(4/8) + 4(2/8) = 3/2, E(Y^2) = 1(4/8) + 4(2/8) = 3/2, \text{Var}(X) = \text{Var}(Y) = 1/2$$

3a-i) put $X = x-10$, then the function becomes $f(X) = -X$ which is odd, $-5 < X < 5$, thus $a_0 = a_n = 0$

$$b_n = \frac{2}{5} \int_0^5 -X \sin\left(\frac{n\pi}{5}\right) X dX = -\frac{2}{5} \left[X(-\cos\left(\frac{n\pi}{5}\right)X) \frac{5}{n\pi} - (-\sin\left(\frac{n\pi}{5}\right)X) \frac{25}{n^2\pi^2} \right] \Big|_0^5 = \frac{10}{n\pi} \cos(n\pi)$$

$$, \text{ therefore } f(x) = \sum_{n=1}^{\infty} \frac{10 \cos(n\pi)}{n\pi} \sin\left(\frac{n\pi}{5}\right)(x-10).$$

3a-ii) This function is even cosine harmonic, therefore $a_0 = \frac{4}{T} \int_0^{T/2} f(x) dx =$

$$\frac{4}{\pi} \int_0^{\pi/2} \cos(x) dx = \frac{4}{\pi}$$

$$a_{2n} = \frac{4}{T} \int_0^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx = \frac{4}{\pi} \int_0^{\pi/2} \cos(x) \cos(2nx) dx = \frac{4\cos(n\pi)}{\pi(2n-1)(2n+1)},$$

$$b_{2n} = 0$$

3-b) Since $T = 2$, therefore $c_n = \frac{1}{2T} \int_{-T}^T f(x) e^{-i\left(\frac{n\pi x}{T}\right)} dx = \frac{1}{4} \int_{-2}^2 e^{-x} e^{-i\left(\frac{n\pi x}{2}\right)} dx =$

$$\frac{1}{4} \int_{-2}^2 e^{-(1+\frac{i n \pi}{2})x} dx = -\frac{1}{2(2+i n \pi)} \left(e^{-(1+\frac{i n \pi}{2})x} \right)_{-2}^2$$

$$= \frac{1}{2(2+i n \pi)} (e^{(2+i n \pi)} - e^{-(2+i n \pi)}) = \frac{i}{(2+i n \pi)} \sin(2+i n \pi)$$

$$= -\frac{1}{(2+i n \pi)} (\cos 2 \sinh n\pi - i \sin 2 \cosh n\pi)$$

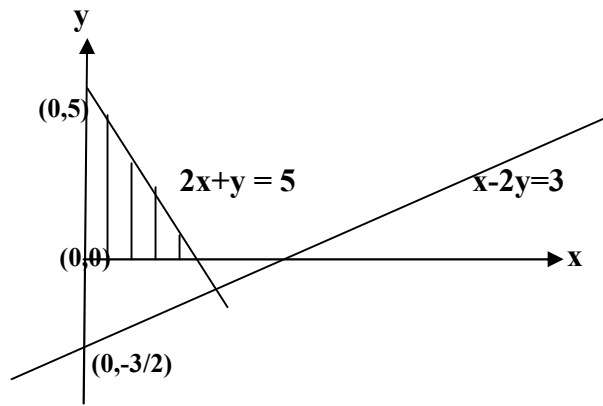
4-a) Since $F_s = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\alpha x) dx = \sqrt{\frac{2}{\pi}} \begin{cases} 1 & 0 \leq \alpha < 1 \\ 2 & 1 \leq \alpha < 2 \\ 0 & 2 \leq \alpha \end{cases}$, and Fourier

integral is expressed by: $f(x) =$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty F_s \sin(\alpha x) d\alpha = \sqrt{\frac{2}{\pi}} \left[\int_0^1 \sqrt{\frac{2}{\pi}} \sin(\alpha x) d\alpha + \int_1^2 2\sqrt{\frac{2}{\pi}} \sin(\alpha x) d\alpha \right] =$$

$$\frac{2}{\pi x} [1 + \cos x - 2\cos 2x]$$

4-b)



$$z_{\max} = 5 \text{ at } (5/2, 0)$$

Dual problem is $L_{\min} = 3p + 5q$ s.t. $p+2q > 2$, $-2p+q > -3$

2nd Method

Maximize $-z + 2x - 3y = 0$, $x - 2y + p = 3$, $2x + y + q = 5$

	x	y	p	q	Const.
p	1	-2	1	0	3
q	2	1	0	1	5
-z	2	-3	0	0	0

	x	y	p	q	Const.
p	0	-5/2	1	-1/2	1/2
x	1	1/2	0	1/2	5/2
-z	0	-4	0	-1	-5

$$z = 5 \text{ at } (5/2, 0)$$



- Answer all the following question
- Illustrate your answers with sketches when necessary.
- No. of questions: 4
- Total Mark: 80 Marks

1-a) A box contains 7 blue , 8 white and 9 red balls, two balls are drawn without replacement. Let X is the number of blue balls and Y is the number of red balls, Find joint probability function, Cov(x,y), P(X+Y=2). **(10 marks)**

1-b) A coin is biased so that heads is twice the tails for three independent tosses of the coin, find

- (i) The probability of getting at most two heads.
(ii) C.d.f. of the r.v. X, and use it to find $P(1 < X \leq 3)$; $P(X > 2)$. **(10 marks)**

2-a) Let X and Y denote the amplitude of noise signals at two antennas. The

random vector (X, Y) has the joint pdf $f(x, y) = ax e^{-ax^2/2} by e^{-by^2/2}$ $x > 0$, $y > 0$, $a > 0$, $b > 0$, find (i) $P[X > Y]$ (ii) Standard deviation of X. **(10 marks)**

2-b) Evaluate m.g.f. for the random variable of exponential and gamma distributions, then deduce μ'_r , $r = 0, 1, 2$ **(10 marks)**

3-a) Expand in fourier series the following periodic functions:

- i) $f(x) = x|X|$, $-1 < x < 1$, ii) $f(x) = x^2$, $0 < x < 2$ **(12 marks)**

3-b) Find complex Fourier for $f(x) = e^{-x}$, $-2 < x < 2$ **(8 marks)**

4-a) Find Fourier transform and Fourier integral $f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$ **(10 marks)**

4-b) Solve the L.P.P. that minimize $z = x-3y$, s.t. $x-2y > 3$, $2x+y > 5$, $x, y > 0$ using Simplex method. **(10 marks)**

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Model answer

1-a)

X \ Y	0	1	2	$f_1(x)$
0	$P(WW) = 0.1014$	$2P(BW) = 0.2029$	$P(BB) = 0.0761$	0.3804
1	$2P(RW) = 0.2609$	$2P(BR) = 0.2283$	0	0.4892
2	$P(RR) = 0.1304$	0	0	0.1304
$f_1(x)$	0.4927	0.4312	0.0761	1

$$P(X+Y=2) = f(1,1) + f(2,0) + f(0,2) = 0.2283 + 0.0761 + 0.1304 = 0.4348$$

$$E(Y) = 0(0.3804) + 1(0.4892) + 2(0.1304) = 0.75, \quad E(X) = 0(0.4927) + 1(0.4312) + 2(0.0761) = 0.5834, \quad E(XY) = 0(0.1014) + 0(0.2029) + 0(0.0761) + 0(0.2609) + 1(0.2283) + 2(0) + 0(0.1304) + 2(0) + 4(0) = 0.2283, \\ \text{therefore } \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = -0.2093$$

1-b) $P(H) = 2 P(T)$, therefore $P(H) = 2/3 = P$, and $P(H \leq 2) =$

$$\sum_{x=0}^2 {}^3c_x (2/3)^x (1/3)^{3-x}$$

$$\text{ii) } F(x=0) = {}^3c_0 (2/3)^0 (1/3)^3, \quad F(x=1) = \sum_{x=0}^1 {}^3c_x (2/3)^x (1/3)^{3-x}, \quad F(x=2) =$$

$$\sum_{x=0}^2 {}^3c_x (2/3)^x (1/3)^{3-x},$$

$$F(x=3) = \sum_{x=0}^3 {}^3c_x (2/3)^x (1/3)^{3-x}, \quad P(1 < X \leq 3) = F(x=3) - F(x=1) =$$

$$\sum_{x=0}^3 {}^3c_x (2/3)^x (1/3)^{3-x} - \sum_{x=0}^1 {}^3c_x (2/3)^x (1/3)^{3-x}, \quad P(X > 2) = F(x=3) - F(x=2) =$$

$$\sum_{x=0}^3 {}^3c_x (2/3)^x (1/3)^{3-x} - \sum_{x=0}^2 {}^3c_x (2/3)^x (1/3)^{3-x}$$

2-a) $P(X > Y)$

$$\begin{aligned}
 &= \int_0^{\infty} \int_y^{\infty} ax e^{-ax^2/2} b y e^{-by^2/2} dx dy = \int_0^{\infty} -e^{-ax^2/2} \Big|_y^{\infty} b y e^{-by^2/2} dy = \int_0^{\infty} e^{-ay^2/2} b y e^{-by^2/2} dy \\
 &= \int_0^{\infty} b y e^{-(a+b)y^2/2} dy = -\frac{b}{a+b} e^{-(a+b)y^2/2} \Big|_0^{\infty} = \frac{b}{a+b}, f_1(x) = \\
 &\int_0^{\infty} [ax e^{-ax^2/2} b y e^{-by^2/2}] dy, \quad E(X) = \int_0^{\infty} x f_1(x) dx \quad \text{and } E(X^2) = \\
 &\int_0^{\infty} x^2 f_1(x) dx, \quad \text{therefore } \text{Var}(X) = E(X^2) - [E(X)]^2
 \end{aligned}$$

2-b) The moment generating function of a exponential distribution is expressed by

$$\begin{aligned}
 E(e^{tx}) &= \int_0^{\infty} e^{tx} (\lambda e^{-\lambda x}) dx = \int_0^{\infty} \lambda e^{-(\lambda-t)x} dx = \frac{\lambda}{(\lambda-t)}, \\
 \mu'_0 &= 1, \mu'_1 = E(X) = \frac{1}{\lambda}, \mu'_2 = E(X^2) = \frac{2}{\lambda^2}
 \end{aligned}$$

The moment generating function of gamma distribution can be expressed by

$$E(e^{tx}) = \int_0^{\infty} e^{tx} \left(\frac{\beta^\alpha}{\Gamma \alpha} x^{\alpha-1} e^{-\beta x} \right) dx = \frac{\beta^\alpha}{\Gamma \alpha} \int_0^{\infty} x^{\alpha-1} e^{-(\beta-t)x} dx$$

$$\text{Put } (\beta-t)x = y \Rightarrow dx = \frac{dy}{\beta-t}, \text{ thus } E(e^{tx}) = \frac{\beta^\alpha}{(\beta-t)^\alpha \Gamma \alpha} \int_0^{\infty} y^{\alpha-1} e^{-y} dy = \frac{\beta^\alpha}{(\beta-t)^\alpha}$$

3-a) The functions $f(x) = x|x|$ is even, therefore $a_0 = a_n = 0$, $T = 1$, therefore

$$b_n = \frac{2}{1} \int_0^1 x^2 \sin\left(\frac{n\pi x}{1}\right) dx =$$

$$\left[x^2 \left(-\frac{1}{n\pi} \cos\left(\frac{n\pi x}{1}\right) \right) - 2x \left(-\frac{1}{n^2 \pi^2} \sin\left(\frac{n\pi x}{1}\right) \right) + 2 \left(-\frac{1}{n^3 \pi^3} \cos\left(\frac{n\pi x}{1}\right) \right) \right]_0^1$$

$$= \left[\left(-\frac{1}{n\pi} \cos\left(\frac{n\pi}{1}\right) \right) + 2\left(\frac{1}{n^3\pi^3} \cos\left(\frac{n\pi}{1}\right) - \frac{1}{n^3\pi^3} \right) \right], \quad \text{thus} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right) =$$

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

This function is $f(x) = x^2$ neither even nor odd, therefore

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right), \quad T=1$$

$$a_0 = \frac{1}{1} \int_0^2 f(x) dx = \frac{1}{1} \int_0^2 (x^2) dx = \left(\frac{x^3}{3} \right) \Big|_0^2 = \frac{8}{3}$$

$$a_n = \frac{1}{1} \int_0^2 (x^2) \cos\left(\frac{n\pi x}{1}\right) dx = \left[(x^2) \left(\frac{1}{n\pi} \sin\left(\frac{n\pi x}{1}\right) \right) - (2x) \left(-\frac{1}{n^2\pi^2} \cos\left(\frac{n\pi x}{1}\right) \right) \right. \\ \left. + 2 \left(-\frac{1}{n^3\pi^3} \sin\left(\frac{n\pi x}{1}\right) \right) \right] \Big|_0^2 = \frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{1} \int_0^2 (x^2) \sin\left(\frac{n\pi x}{1}\right) dx = \left[(x^2) \left(-\frac{1}{n\pi} \cos\left(\frac{n\pi x}{1}\right) \right) - (2x) \left(-\frac{1}{n^2\pi^2} \sin\left(\frac{n\pi x}{1}\right) \right) \right. \\ \left. + 2 \left(\frac{1}{n^3\pi^3} \cos\left(\frac{n\pi x}{1}\right) \right) \right] \Big|_0^2 = \frac{-4}{n\pi}$$

$$3-b) \text{ Since } T = 2, \text{ therefore } c_n = \frac{1}{2T} \int_{-T}^T f(x) e^{-i\left(\frac{n\pi x}{T}\right)} dx = \frac{1}{4} \int_{-2}^2 e^{-x} e^{-i\left(\frac{n\pi x}{2}\right)} dx$$

$$= \frac{1}{4} \int_{-2}^2 e^{-(1+\frac{i n \pi}{2})x} dx = -\frac{1}{2(2+i n \pi)} \left(e^{-(1+\frac{i n \pi}{2})x} \right) \Big|_{-2}^2$$

$$= \frac{1}{2(2+i n \pi)} (e^{(2+i n \pi)} - e^{-(2+i n \pi)}) = \frac{i}{(2+i n \pi)} \sin(2+i n \pi)$$

$$= -\frac{1}{(2+i n \pi)} (\cos 2 \sinh n\pi - i \sin 2 \cosh n\pi)$$

4-a) Since this function is even, therefore there is only Fourier Cosine transform

$$\text{such that } F_c(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x \, dx = \sqrt{2/\pi} \int_0^1 (1-x^2) \cos \alpha x \, dx$$

$$= \sqrt{2/\pi} \left[(1-x^2) \left(\frac{\sin \alpha x}{\alpha} \right) - (-2x) \left(\frac{-\cos \alpha x}{\alpha^2} \right) + (-2) \left(\frac{-\sin \alpha x}{\alpha^3} \right) \right]_0^1$$

$$= \sqrt{2/\pi} \left[\frac{-2 \cos \alpha}{\alpha^2} + \frac{2 \sin \alpha}{\alpha^3} \right] = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin \alpha - \alpha \cos \alpha}{\alpha^3} \right]$$

Therefore Fourier integral $f(x)$ is expressed by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\alpha) \cos \alpha x \, d\alpha$

4-b) We can solve it using dual problem such that $L_{\max} = 3p + 5q$, s.t. $p + 2q < 1$, $-2p + q < -3$,

therefore $2p - q > 3$ so that $p + 2q + s = 1$, $2p - q - t + u = 3$, where s is slack variable, t is surplus and u is artificial such that $w = u \Rightarrow w - 3 = -2p + q + t$

	p	q	s	t	u	Const.
s	1	2	1	0	0	1
u	2	-1	0	-1	1	3
-L	3	5	0	0	0	0
-w	-2	1	0	1	0	-3

	p	q	s	t	u	Const.
p	1	2	1	0	0	1
u	0	-5	-2	-1	1	1
-L	0	-1	-3	0	0	-3
-w	0	5	2	1	0	-1

Since the row of w has only positive elements or zeros, therefore there is no solution